PASCUAL JORDAN

FINAL REPORT I, 1961

# THE MATHEMATICAL THEORY OF QUASI ORDER, SEMI GROUPS OF IDEMPOTENTS AND NONCOMMUTATIVE LATTICES – A NEW FIELD OF MODERN ALGEBRA

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## About this Report.

The theory of skew lattices - a new chapter (or a new paragraph) of abstract algebra - is discussed here in such a manner, that 1) the greater part of known important results concerning this field is covered here; 2) no knowledge of the reader concerning already published parts of the theory is needed in order to understand what is said here.

Many of the details discussed here are already published in articles of the author, partly together with E. Witt and W. Böge. But only in this report the systematical trend of the new mathematical theory is clearly to be seen - so that the details find their appropriate frame. At the same time many proofs could be simplified considerably after the connections of the whole matter have been stepwise better understood - many details of the results, originally found by highly complicated considerations, at last could be proved in a very short and simple manner.

This process of concentration in the development of the theory allowed also a strong reduction of the length of this presentation of the theory. Additionally this length has been limited by omitting much material which to discuss here would have lead to far. In my mentioned papers as well as in unpublished manuscripts many further details are contained which till now did not allow to discern their systematical significance - these many still isolated statements may be reserved for further study. But also to evaluate and use the beautiful ideas, concerning our topic, developed by S. Matsushita, is a task not yet accomplished.

Naturally a considerable part of the theorems presented in this report here are new ones, not yet published anywhere. Several meaningful contributions to the theory made by W. Böge, to whom I am very much indebted indeed, could be included here.

Especially Lemma 16 and lemma 17, given by Böge, show how and why the new theory of skew lattices must be acknowledged as a necessary and unavoidable part of mathematical research.

## CHAPTER I. THE CONCEPT OF SKEW LATTICES

§ 1. The mathematical theory of skew lattices - a new branch of abstract algebra - is a generalisation of the well known theory of lattices. Taking instead of the two commutative operations of the lattice theory two operations which must not be commutative, this new theory deviates from the usual lattice theory in a similar manner as the general theory of groups deviates from the theory of abelian groups: The theory of skew lattices is more complicated and more difficult, but also much rich and more interesting than the theory of lattices.
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Groups as well as lattices occur in almost every chapter of mathematics, and their theory therefore is an indispensable tool of nearly all branches of mathematics. Skew lattices are not so common - examples of these ust be detected or constructed instead of being seen at once in many mathematical problems. But great varieties of skew lattices do actually exist, and especially many of these arise from the study of lattices. Therefore the theory of skew lattices is not only a generalisation of the theory of lattices but to a certain extent also a part of this theory.

Close connections exist between the theory of skew lattices and the theory of semi groups. Especially the mathematical theory of those semigroups which contain only idempotent elements, is an essential part of the theory of skew lattices. But also other types of semi groups occur in the frame of the theory of skew lattices.

Definition: A set of elements $a, b, \ldots$ is a skew lattice, if from each ordered pair of elements $a, b$ two compositions of new elements $a_\wedge b$ and $b_\vee a$ can be made by operations $\wedge, \vee$ fulfilling the following axioms:

(1)(A) $\left\{\begin{array}{l} (a_\wedge b)_\wedge c = a_\wedge(b_\wedge c), \\ (a_\vee b)_\vee c = a_\vee(b_\vee c); \end{array}\right.$

(2)(B) $(a_\wedge b)_\vee a = a_\wedge(b_\vee a) = a$ .

Therefore those skew lattices which are commutative with respect to each one of the two operations $\wedge, \vee$ are the common lattices.

Instead of the signs $\wedge, \vee$ we use often the signs $\cdot, +$ of multiplication and addition.

As a consequence of (2) - even without using the associativity (1) - we get

$$a \wedge a = a \vee a = a \ . \tag{3}$$

All elements of any skew lattice are multiplicative and additive idempotents.

Therefore a skew lattice $W$ is a semi group of idempotents with respect to addition and to multiplication. We shall see later that every semigroup of idempotents does occur as the multiplicative or additional semigroup of certain skew lattices.

Principle of duality: The axioms (1),(2) remain invariant if one 1) permutes the operations $\wedge, \vee$ ; 2) reads every line from behind.

Definition: The skew lattice $W$ is a skew lattice with orthogonality, if there exists to each element $a$ an element $\bar{a}$ so that the following axioms are fulfilled:

$$(4) \quad \begin{cases} \bar{\bar{a}} = a \ , \\ \overline{a \wedge c} = \bar{c} \vee \bar{a} \ . \end{cases}$$

We have then from (2):

$$\bar{a} \wedge \overline{a \wedge c} = \bar{a} \ . \tag{5}$$

Lemma 1: If in a (multiplicative) semi group $H$ of idempotents an involutory relation $a \to \bar{a}$ fulfilling (5) exists, then the elements of $H$ form a skew lattice, if the second operation (addition) is defined by the second line of (4).

The possibility of a non commutative generalisation of the theory of skew lattices has been emphasized at first by F. Klein-Barmen, who studied in this connection the free semi group of idempotents with two generating elements. A systematical study of skew lattices has been started by the author of this report, partly in collaboration

with E. Witt and W. Böge who made important contributions to this enterprise. Independently of this author S. Matsushita studied the non commutative generalisation of lattices. The following is a complete list of the present literature of this topic:

P. Jordan:

1) Über nichtkommutative Verbände
Arch. Math. 1,56 (1949)
2) Zur Quanten-Logik. Arch. Math. 2, 166 (1949/50)
3) Zum Dedekindschen Axiom in der Theorie der Verbände
Math. Sem. Hamburg 16, 71 (1949)
4) Algebraische Betrachtungen zur Theorie des Wirkungsquantums und der Elementarlänge. Math. Sem. Hamburg 18, 99, (1952)
5) Zur Theorie der nichtkommutativen Verbände
Akad. Main 1952, S. 61
6) Bericht über die nichtkommutativen Verbände.
Festschrift für B. Kraft. 1954. S. 551.
7) Beiträge zur Theorie der Schrägverbände
Akad. Mainz 1956, S. 29
8) Über distributive Schrägverbände
Akad. Mainz 1958, S. 229
9) Quantenlogik und das kommutative Gesetz
Sympos. Axiom. Method (1960), 365
10) Über nichtkommutative Verbände
Celebra ione di Archimede del XX. Secolo (in print).
11) Über distributiv-modulare Schrägverbände
Akad.Mainz (in print).
12) P. Jordan u. E. Witt, Zur Theorie der Schrägverbände.
Akad. Mainz 1953, S. 225.
13) P. Jordan u. W. Böge, Zur Theorie der Schrägverbände II.
Akad. Mainz 1954, S. 79
14) F. Klein-Barmen, Über eine weitere Verallgemeinerung des Verbands=begriffes. Math. ZS. 46, 472 (1940)
15) F. Klein-Barmen, Ordoid, Halbverband und ordoide Semigruppe
Math. Annalen 135, 142 (1958)
16) S. Matsushita, Lattices non commutatifs.
C.R. 1953, S. 1526 (1953)
17) S. Matsushita, Ideal in non-commutative lattices.
Proc. Japan Acad. 34, 407, (1958)
18) S. Matsushita, Zur Theorie der nichtkommutativen Verbände I.
Math. Annalen 137,1 (1959)
19) I.A. Green and D. Rees, On semi groups in which $x^r = x$.
Proc. Camb. Phil. Soc. 48, 35 (1952).

## CHAPTER II. SEMI GROUPS OF IDEMPOTENTS

§ 2. Definition: A semigroup of idempotents may be called a half skew lattice HSL.

In the following we write the half skew lattices as multiplicative semi half groups, denoting the product of $x$ and $y$ by $x.y$ or by $xy$. But the reader may please take in mind: If later we apply the results of our discussion in this chapter to skew lattices, we shall interpret $xy$

as $x_\wedge y$ in the case of the $\wedge$-HSL in any W,

as $y_\vee x$ in the case of the $\vee$-HSL in any W.

Special classes of HSL are defined by additional axioms. We mention the following examples of such axioms defining several important classes:

Commutativity

(6) $ab = ba$ ;

"Halfnest":

(7) $ab = a$ ;

"Antihalfnest":

(7,1) $ab = b$ .

"Superflat HSL":

(8) $abc = acb$ ;

"Flat HSL":

(9) $aba = ab$ ;

Without special names:

(10) $aba = a$ ;

(11) $abc = ac$ ;

(12) $abac = abc$ ;

(13) $caba = cba$ ;

(14) $abcd = acbd$ ;

(15) $abaca = abca$ .

Obviously (8) is a weaker consequence as well of (6) as of (7);

and (9) a weaker consequence of (8). The axioms (10) and (11) are equivalent; for as consequence of (10) we get:

$$abc = ab(ac)bc = aba.cbc = ac. \tag{16}$$

The axioms (12), (13) and (15) too are consequences of (14).

The axiom (7,1) has a totally different meaning from (7) in the frame of the theory of skew lattices - owing to what has been said above about the interpretation of $xy$ as $x_{\wedge}y$ or $y_{\vee}x$ - though in the frame of a theory of semi groups of idempotents (7) and (7,1) are entirely symmetrical.

Fulfilment of the equation

$$ab = a \tag{17}$$

by two special elements $a, b$ may be called an inclusion. In the case $a_{\wedge}b = a$ we say that the element $a$ is included in $b$; in the case $b_{\vee}a = a$ we say that $b$ is included in $a$. In both cases this inclusion is transitive in consequence of (1), and reflexive in consequence of (2).

The same remarks are to be made about another inclusion, defined by

$$ba = a\ . \tag{18}$$

We call the case (17) weak inclusion, and (18) strong inclusion.

Lemma 3: In any half skew lattice the halfnests are the equivalence classes of weak inclusion; the antihalfnests are the equivalence classes of strong inclusion.

Lemma 4: In any HSL the following three properties are equivalent:

A) There exists no antihalfnest with more than one element;

B) weak inclusion is a consequence of strong inclusion;

C) axiom (9) holds.

Proof: From C) we have B), that is $ab = a$ as consequence of $ba = a$. From B) we have A), that is $a = b$ as consequence of $ba = a$ and $ab = b$. From B) we have also C): $ba = a \rightarrow ab = a$ gives $xyx = xy$ in the case $a = xy, b = x$. From A) we get C): $ab = b$, $ba = a \rightarrow a = b$ gives $xyx = xy$ in the case $a = xy$, $b = xyx$.

Lemma 5: In any flat HSL the halfnests are a system of congruence classes.

Proof: If $a, a'$ is a halfnest, then in the flat case also the pairs of elements $ab, a'b$ and $ca, ca'$ are halfnests. For we have $aba'b = aa'ba'b = aa'b = ab$ and $caca' = caa'ca' = caa' = ca$.

Lemma 6: The commutative HSL are those in which weak and strong inclusions coincide.

Proof: From lemma 4 and its proof we see: If weak inclusion is a consequence of strong inclusion, then we have $aba = ab$. If strong inclusion is a consequence of weak inclusion, we have $aba = ba$. -

From two HSL's $H_o$ and $H_1$ with elements $a_o, b_o, \ldots$ and $a_1, b_1, \ldots$ we can derive a new HSL called by definition the chain composition $(H_o, H_1)$ of $H_o$ and $H_1$. Its elements are those of $H_o$ together with those of $H_1$, so that $H_o$ and $H_1$ are subsystems of $(H_o, H_1)$; the composition of any element $a_o$ of $H_o$ with any element of $H_1$ being given by

$$a_o a_1 = a_1 a_o = a_o \,. \tag{19}$$

Definition: An axiom characterising a certain class of HSL's is called conservative if its validity for $H_o$ and $H_1$ causes also its validity for the chain composition $(H_o, H_1)$.

Lemma 7: The axioms (6), (9), (15) are conservative ones.

§ 3. We now proceed to determine for some of the classes of HSL defined by the additional axioms above the free system with $n$ generating elements $a_1, a_2, \dots, a_n$.

1) In the case of a halfnest (or antihalfnest) the generating elements $a_k$ are the only ones.

2) In the case of axiom (11) each element of the free system may be written as

(20) $$a_{kl} = a_k a_l$$

with

(21) $$a_{kl} a_{hj} = a_{kj}.$$

If we now take $n^2$ elements $a_{kl}$ and define their composition by (21), then we see, that this definition fulfils associativity, idempotency $a_{kl} a_{kl} = a_{kl}$, and the additional axiom (11). Therefore these are $n^2$ different elements of the free system.

3) In the case of axiom (8) - superflat HSL - each element can be written as

(22) $$a = a_{k_0} a_{k_1} \dots a_{k_m}$$

with $m$ different index values $k_0, k_1, \dots k_m \leq n$.

Let us use the symbol

(23) $$a = (k_o, K),$$

where $K$ is the set of values $k_o, k_1, \dots k_m$.

Composition ist obviously given by

(24) $$aa' = (k_o, K \cup K').$$

Taking now (24) as definition of the composition of symbols (23), we see that this composition gives a HSL and fulfils the additional axiom (8). Therefore the

(25 $$U(n) = n2^{n-1}$$

different symbols (23) are indeed $U(n)$ different elements of the free system.

4) In the case of flat HSL's, axiom (9), the general element again can be written as (22); we now use the notation

(26) $$a = (k_o k_1 \dots k_m);$$

the composition is defined by

(27) $$aa' = (k_o k_1 \dots k_m k'_o k'_1 \dots k'_{m'})$$

with the additional remark that all those $k'_r$ are to be omitted afterwards which equal any of the numbers $k_s$.

Taking again this as definition of the composition of symbols (26), we get a HSL, fulfilling (9), and therefore the

(28) $$G(n) = n! \sum_{k=o}^{n-1} \frac{1}{k!}$$

different symbols (26) are different elements of the free system.

5) In the case of axiom (15) let us consider the elements

(29) $$a = a_{k_o} a_{k_1} \dots a_{k_m} a_{h_o} a_{h_1} \dots a_{h_m},$$

whre all $k_o, k_1, \dots, k_m$ are different, and the $h_o, h_1, \dots, h_m$ are any permutation of the $k_r$. We denote (29) by the symbol

(30) $$a = (k_o k_1 \dots k_m | h_o h_1 \dots h_m);$$

we have then especially

(31) $$a_k = (k|k).$$

From (15) we get the following composition rule: We have to write down

(32) $$aa' = (k_o \dots k_m k'_o \dots k'_{m'} | h_o \dots h_m h'_o \dots h'_{m'}),$$

and afterwards to omit the common index values of $a$ and $a'$ among the $k'_j$ and also among the $h_j$ .

To prove this rule we write, using (15);

$$aa' = aa'aa' \tag{33}$$
$$= a_{k_0} \cdots a_{k_0'} \cdots a_{h_0} \cdots a_{h_0'} \cdots ,$$

so that (32) is justified; and the rest in the formulation of our rule comes too from (15).

Again taking now the symbols (31) as elements, and our rule as definition of their composition, we get a HSL , fulfilling (15). Therefore the

$$P(n) = n! \sum_{m=0}^{n-1} \frac{(n-m)!}{m!} \tag{34}$$

different symbols (31) correspond with the different elements of the <u>free</u> system. We have $P(2) = 6$; $P(3) = 51$.

§ 4. <u>Definition.</u> A half skew lattice is called an <u>ordered</u> one if it ~~it~~ fulfils the axiom

$$ab = a \text{ OR } b , \tag{35}$$

so that each pair a,b of its elements is a sub system.

Therefore each pair a,b of elements in an ordered HSL must correspond to one of the following four possibilities:

(36)
1) a,b form a halfnest;
2) a,b form an antihalfnest;
3) a is twofold included in b;
4) b is twofold included in a.

There are these four possibilities only, because we have for ab and for ba two possibilities a and b.

<u>Lemma 8. The ordered HSL's are the chains of halfnests and antihalfnests.</u>

<u>Proof:</u> Any element x in an ordered HSL cannot belong

to a halfnest as well as to an antihalfnest both of more than one element. If $x, y$ form a halfnest, and $x, z$ an antihalfnest:

$$(37) \quad \begin{cases} xy = zx = x, \\ yx = y, \\ xz = z, \end{cases}$$

this together with

$$(38) \qquad zy = yz = y$$

would lead to $xy = y = x$ ; and (37) together with

$$(39) \qquad zy = yz = z$$

would lead to $yx = x = y$

We denote now any finite ordered HSL by a symbol as

$$(40) \quad \begin{cases} H = {}_\wedge(n_1, n_2^*, \ldots , n_r) \\ = (H^{(1)}, A^{(2)}, \ldots , H^{(r)}), \end{cases}$$

meaning a chain composition, containing a halfnest of $n_1$ elements (all its elements are weakly and strongly included in all other elements of $H$), an antihalfnest of $n_2$ elements, and so on.

For example

$$(41) \qquad H = {}_\wedge(3,1)$$

is a flat HSL with 4 elements, which may be denoted here as $0, u, v, 1,$ with the following compositions:

$$(42) \quad \begin{cases} 0_\wedge x = 0, \\ u_\wedge x = u, \\ v_\wedge x = v, \\ 1_\wedge x = x. \end{cases}$$

With $x$ we are denoting here the general element of $H_4$.

This example may be used to show how lemma 1 works. We define

in $H_4$ an involutional correspondence $x \to \bar{x}$ by:

$$(43) \qquad \begin{cases} \bar{0} = 1, \; \bar{1} = 0, \\ \bar{u} = u, \; \bar{v} = v \,. \end{cases}$$

The condition (5) obviously is fulfilled; for in each case at least one of the elements $a, \bar{a}$ belongs to the elements $z$ with the property $z_\wedge x = z$. Therefore we get a <u>skew lattice</u> $W_4$ (with orthogonality):

$$(44) \qquad \left\{ \begin{array}{l|l} 0_\wedge x = 0 & u_\vee 1 = 1 \\ u_\wedge x = u & x_\vee u = u \\ v_\wedge x = v & x_\vee v = v \\ 1_\wedge x = x & x_\vee 0 = x \,. \end{array} \right.$$

## CHAPTER III. BASIC LAWS OF SKEW LATTICES

§ 5. Let $W$ be a set of elements who form in <u>two</u> ways a semi group of idempotents; one of these compositions being denoted by $\wedge$ , the other one by $\vee$ .

<u>Under what conditions will this system be a skew lattice, fulfilling (2)?</u>

At first we see from (2) that in every skew lattice strong multiplicative (additive) inclusion of the element $a$ in $b$ has as its consequence weak additive (multiplicative) inclusion of $a$ in $b$ . This may be expressed by the graphical scheme:

(45)

| | | |
|---|---|---|
| strong inclusion: | $b_\wedge a = a$ | $b_\vee a = b$ |
| weak inclusion | $a_\vee b = b$ | $a_\wedge b = a$ |

;

$$(46) \qquad \begin{array}{|c|c|} \hline \downarrow & \downarrow \\ \hline \end{array} \,.$$

Only one of these two statements needs a proof: From

(47) $$(b_{\wedge} a)_{\vee} b = b$$

we see, that $b_{\wedge} a = a$ has the consequence $a_{\vee} b = b$. The other statement is dual to this one.

But (46) gives not only a necessary, but also a sufficient condition for $W$ being a skew lattice. For the element $b_{\wedge} a$ is multiplicatively strongly included in $b$ ; therefore according to (46) it must also be additively weakly included in $b$, as expressed by (47).

Lemma 9: Any set $W$ of elements forming a multiplicative (operation $\wedge$ ) and at the same time an additive (operation $\vee$ ) semi group of idempotents is a skew lattice if and only if each case of strong inclusion is connected with weak inclusion of the other kind (multiplicative or additive).

Several special cases may be considered:

Lemma 10: If $W$ is a multiplicative (additive) half nest and any arbitrary additive (multiplicative) HSL, then $W$ is a skew lattice.

Lemma 11: If any skew lattice is a multiplicative (additive) antihalfnest, then it is an additive (multiplicative) halfnest.

Definition: A skew lattice being a multiplicative and additive halfnest is called a nest.

Lemma 12: The nests are the equivalence classes of both multiplicative and additive weak inclusion.

Lemma 13: Each equivalence class of both multiplicative and additive strong inclusion contains only one element.

This is a consequence of lemma 11.-

The nests are those skew lattices which fulfil the axiom

(48) $$a_{\wedge} b = b_{\vee} a \text{ .}$$

With (48) we get from (48) that $a_{\wedge} b = a$.

The other axiom

(49) $$a_\wedge b = a_\vee b$$

is valid only in skew lattices with elements $a_{l\lambda}$ and

(50) $$a_{l\lambda} \wedge a_{m\mu} = a_{l\mu} .$$

For (49) and (2) lead to (10) and therefore to (11) and to (20), (21), a special case of (50). The general case of finite skew lattices fulfilling (49) can be derived from the <u>free</u> systems (20), (21) by <u>congruence</u> relations; and congruence classes in a skew lattice of type (50) give skew lattices of this same type.

<u>Proof:</u> Let be $a_{l\lambda} \equiv a_{j\iota}$, where $l \neq j$. Then we have from (50):

(51) $$\left\{ \begin{aligned} & a_{l\lambda} \wedge a_{m\mu} \equiv a_{j\iota} \wedge a_{m\mu}, \\ & a_{l\mu} \equiv a_{j\mu} . \end{aligned} \right.$$

Therefore in the system of congruence classes all $a_{l\mu}$ can be replaced by the corresponding $a_{j\mu}$.

Obviously the skew lattice (50) is the <u>direct product</u> of a multiplicative antihalfnest ( and therefore additive halfnest ) $\beta_\lambda$ and an additive antihalfnest $\alpha_l$:

(52) $$a_{l\lambda} = \alpha_l \times \beta_\lambda .$$

<u>Lemma 14: The axiom $a_\wedge b = a_\vee b$ is fulfilled only by all direct products of antihalfnests.</u>

<u>Definition:</u> The chain composition $(W_0, W_1)$ of two skew lattices $W_0$, $W_1$, with elements $a_0, b_0, \ldots$ and $a_1, b_1, \ldots$ is that skew lattice which as a multiplicative and additive HSL is chain composition of the corresponding HSL's in $W_0$ and $W_1$:

(53) $$\left\{ \begin{aligned} & a_0 \wedge a_1 = a_1 \wedge a_0 = a_0, \\ & a_0 \vee a_1 = a_1 \vee a_0 = a_1 . \end{aligned} \right.$$

That this indeed is again a skew lattice can be seen too from lemma 9.

<u>Lemma 15: Those elements of a skew lattice which are additively weakly included in a certain element $c$, form a sub system. - The same</u> statement holds for those elements which multiplicatively include weakly $c$.

<u>Proof:</u> From $c_\wedge a = c$; $c_\wedge b = c$ we get not only $c_\wedge a_\wedge b = c$, but also $c_\wedge(a_\vee b) = c_\wedge b_\wedge(a_\vee b) = c_\wedge b = c$.

<u>Definition:</u> Any set $M$ is called a quasi ordered set, if for some pairs of ( unequal or equal ) elements of $M$ a relation $\leq$ is defined in a reflexive and transitive manner. ( Special case: $a \leq a$ for each element, but no other relation exists. Other special case: $a \leq b$ for each pair $a, b$ in $M$ ).

<u>Lemma 16 (W. Böge): If $M$ is a quasi ordered set, and the elements of $M$ are in two ways semigroups - with operations $\wedge, \vee$ - having the following properties:</u>

1) $a_\wedge b \leq a$,
2) $a_\wedge b = a$ in all cases $a \leq b$;
3) $a_\vee b \geq b$,
4) $a_\vee b = b$ in all cases $a \leq b$,

<u>then $M$ is a flat skew lattice. - Every flat skew lattice can be described in this manner.</u>

<u>Proof:</u> I. From 3) we have $b_\vee a \geq a$, therefore from 2): $a_\wedge(b_\vee a) = a$, and from 4): $a_\vee b_\vee a = b_\vee a$. Dually symmetric to these statements are $(a_\wedge b)_\vee a = a$ and $a_\wedge b_\wedge a = a_\wedge b$. - II. In any flat skew lattice we define $a \leq b$ so that it means <u>twofold</u> (multiplicative and additive) inclusion of $a$ in $b$:

$$x \leq y \iff \begin{cases} x_\wedge y = x, \\ x_\vee y = y. \end{cases} \tag{54}$$

This indeed is fulfilled by $x = a_\wedge b, y = a$; therefore this relation (54) indeed has all properties 1),2),3),4).
(The two special cases mentioned above both lead to a <u>nest</u>).
The connexion with lemma 9 and lemma 4 is this one: $a_\wedge b$ is additively strongly included in $a$, and therefore has to be ( in the flat case ) twofold weakly included in $a$.

From this lemma <u>Böge</u> derived the following example of a flat skew lattice: Let $M = \{a, b, \ldots\}$ be the set of all reflexive transitive relations $\subseteq$ in a set $S = \{x, y, \ldots\}$. Any element $a$ of $M$ means that in a certain manner for every pair $x, y$ of elements of $S$ the relation $x \subseteq y$ is given or not given. In the former of these two cases we write $xay$; in the latter case we write $\overline{xay}$.

Now we define in $M$ the relation $\subseteq$ by:

(55) $a \subseteq b$ means $xay \Rightarrow xby$ for every pair $x, y$ in $S$.
This is a reflexive and transitive relation.

Secondly we define $a_\wedge b$ by:

$$(56) \quad x(a_\wedge b)y \Longleftrightarrow xay \text{ AND } \left\{ \begin{array}{l} \overline{yax} \\ \text{OR} \\ yax \text{ AND } xby. \end{array} \right.$$

Using the <u>Boolean distributive lattice</u> of AND and OR, denoting AND,OR by $., +$, we can write (56) also thus:

$$(57) \quad x(a_\wedge b)y = xay.(\overline{yax} + yax.xby) .$$

This is <u>associative.</u>

<u>Proof</u>: We have

$$(58) \left\{ \begin{array}{l} x(a_\wedge(b_\wedge c))y = xay.\ (\overline{yax} + yax.x(b_\wedge c)y) \\ = xay.(\overline{yax} + yax.\ xby.\ (\overline{ybx} + ybx.\ xcy)). \end{array} \right.$$

At the other hand we get:

$$x((a_\wedge b)_\wedge c)y = x(a_\wedge b)y.\ (\overline{y(a_\wedge b)x} + y(a_\wedge b)x.\ xcy)$$

$$= xay.\ (\overline{yax} + yax.\ xby).\ (\overline{y(a_\wedge b)x} + y(a_\wedge b)x.\ xcy);$$

$$\overline{y(a_\wedge b)x} = \overline{yax.\ (\overline{xay} + xay.ybx)}$$

$$= \overline{yax} + xay.(\overline{xay} + \overline{ybx}) = \overline{yax} + xay.\ \overline{ybx};$$

$$x((a_\wedge b)_\wedge c)y$$

$$= xay.\ (\overline{yax} + yax.xby.(xay.\overline{ybx} + y(a_\wedge b)x.xcy))$$

$$= xay.\ (\overline{yax} + yax.\ xby.\ (\overline{ybx} + (\overline{xay} + ybx).xcy))$$

$$= a_\wedge(b_\wedge c).$$

And $\wedge$ has the properties 1),2). For $x(a_\wedge b)y \Rightarrow xay$ according to (56); and $a_\wedge b = a$ as soon as $xay \Rightarrow xby$.

<u>Lemma 17 (W. Böge): The reflexive transitive relations $a,b,\ldots$ in a set of elements $x,y,\ldots$ form a flat skew lattice if their compositions</u> $\wedge, \vee$ <u>are defined by</u> (56), (57) <u>for</u> $\wedge$, <u>and dually for</u> $\vee$ .

This lemma 17 is especially interesting because it shows that at least the theory of <u>flat</u> skew lattices is an <u>unavoidable</u> part of the theory of quasi order.

As the last point in this paragraph we consider the <u>ordered skew lattices</u>, which, by definition, are those which have two ordered HSL's, so that each pair of elements $a,b$ forms a sub skew lattice. (Any HSL of two elements is commutative or a half nest or a antihalfnest. A skew lattice of two elements therefore is a lattice $V_2$ or a nest $N_2$ or a halfcommutative halfnest (look at (65), §6), or an antihalfnest).

We discuss here only <u>finite</u> ordered skew lattices. Owing to the fact that each set of elements of an ordered skew lattice is a sub skew lattice, we can make from the elements a series so that the

following statements are correct, using the denotation from (40):

1) In the flat case the symbol

(59) $$W = {}_{\wedge}(n_1, n_2, \ldots, n_r)\,|\,{}_{\vee}(m_1, m_2, \ldots, m_s)$$

with

(60) $\sum n_j = \sum m_k$ = number of elements ~~means especially~~ means especially that $\min(n_1, m_1)$ elements form a nest of elements strongly included in all other elements, and twofold strongly included in $n - \max(n_1, m_1)$ other elements. Omitting these $\min(n_1, m_1)$ elements there remains a skew lattice $W'$ of $n - \min(n_1, m_1)$ elements, namely in the case $n_1 > m_1$:

(61) $$W' = {}_{\wedge}(n_1 - m_1, n_2, \ldots, n_r)\,|\,{}_{\vee}(m_2, \ldots, m_s);$$

in the case $n_1 > m_1$ :

(62) $$W' = {}_{\wedge}(n_2, \ldots, n_r)\,|\,{}_{\vee}(m_1 - n_1, m_2, \ldots, m_s);$$

in the case $n_1 = m_1$ :

(63) $$W' = {}_{\wedge}(n_2, \ldots, n_r)\,|\,{}_{\vee}(m_2, \ldots, m_s).$$

<u>Lemma 18: The symbol (59) with (60) represents in every case a possible structure of flat ordered skew lattices; and each such structure corresponds to a uniquely determined symbol</u> (59).

<u>Proof using lemma 9:</u> In order to be multiplicatively strongly included in an element $y$ belonging to the multiplicative halfnest (with $n_j$ elements) denoted by $n_j$, and to an additive halfnest denoted by $m_h$, the element $x$ must belong to any $\wedge$-halfnest denoted by $n_l$ with $l < j$; then it belongs to a $\vee$-halfnest denoted by $m_i$ with $i \leq h$.

The general case, allowing also the presence of antihalfnests, can be described by symbols similar to (59), but with asterics Γ at some of the numbers $n_j$, $m_h$ . Allowing also (superfluous) values 0 of these numbers, we can write for the general case:

Γ(meaning antihalfnests)

$$(64) \qquad W = {}_{\wedge}(n_1, n_2^*, n_3, n_4^*, \ldots \qquad , n_r) |_{\vee} (m_1, m_2^*, \ldots, m_s).$$

Lemma 19: The symbol (64) gives an ordered skew lattice if and only if the elements denoted by any $n_j^*$ are entirely contained in those denoted by a certain $m_h$, and vice versa.

§ 6. Definition: An axiom, characterising a class of skew lattices, is called an HN-axiom, if it is fulfilled in the case of every half nest( multiplicative or additive).

The axiom (2) is an HN-axiom according to lemma 10. An example of an axiom which is not an HN-axiom, is the following one, which is fulfilled especially if at least one of the operations $\wedge, \vee$ is commutative:

$$(65) \qquad \begin{cases} (a_\wedge b)_\vee (b_\wedge a) = (b_\wedge a)_\vee (a_\wedge b), \\ (a_\vee b)_\wedge (b_\vee a) = (b_\vee a)_\wedge (a_\vee b). \end{cases}$$

Definition: An axiom for skew lattices is called conservative, if its validity for $W_o$ and $W_1$ guarantees also its validity for the chain composition $(W_o, W_1)$.

The axiom (2) is conservative; the axioms (48) and (49) are not conservative ones.

Definition: A skew lattice is flat if both its HSL's are flat according to axiom (9):

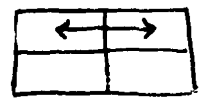
$$(66) \qquad \begin{cases} a_\wedge b_\wedge a = a_\wedge b; \\ a_\vee b_\vee a = a_\vee b. \end{cases}$$

Our former statement that (47) is equivalent with the law that strong multiplicative inclusion has weak additive inclusion as its consequence, can be applied with: Permutation of $x, y$ in $x_\wedge y$; permutation of $x, y$ in $x_\vee y$; permutation of $\wedge, \vee$. Out of the

eight statements arising in this manner, only two have been discussed in § 5. Now we mention also the following three additional dually symmetric axioms (F), (C), (H), containing each one two equations which can be interpreted according to those eight statements:

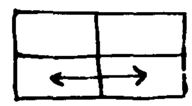
(67) (F) $$a_\vee(a_\wedge b) = (b_\vee a)_\wedge a = a \; ;$$

thus: this axiom can be indicated the meaning of

(F)  .

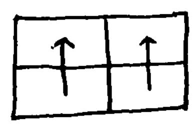
(68) (C) $$a_\wedge(a_\vee b) = (b_\wedge a)_\vee a = a;$$

this axiom means

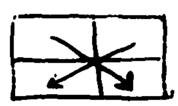
(C)  .

(69) (H) $$a_\vee(b_\wedge a) = (a_\vee b)_\wedge a = a;$$

this means

(H) 

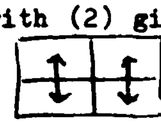
According to lemma 4 the axiom of a flat skew lattice means

(G)  .

From this it is to be seen that (G) for both operations $\wedge, \vee$ is a consequence as well of (F) as of (C); for in all skew lattices (46) is valid.

Lemma 20: A skew lattice fulfilling one of the axioms (C), (F) is a flat one.

Combination of (H) with (2) gives

 ;

we see, that in the flat case (H) guarantees commutativity.

The axiom (C) is fulfilled already if

(70) $$a_\wedge(a_\vee b) = (b_\wedge a)_\vee a$$

is valid - : we then have the consequence $(b_\wedge a)_\vee a = a_\wedge[(b_\wedge a)_\vee a] = a$.
But the axiom

(71) $$a_\vee(b_\wedge a) = (a_\vee b)_\wedge a$$

is weaker than (69). This axiom (69) obviously is fulfilled in the case (49) and in the commutative case. Other examples are not yet known.

All axioms written down above in this paragraph, (65) till (71), are conservative ones; but among them only (71) is an HN-axiom.

A further example of a conservative HN-axiom is this:

(72) $$\begin{cases} a_\vee(a_\wedge b) = a_\wedge(a_\vee b), \\ (b_\vee a)_\wedge a = (b_\wedge a)_\vee a, \end{cases}$$

valid especially in the case that (C) and (F) both are fulfilled,

From lemma 5 we get now

Lemma 21: In a skew lattice fulfilling the axiom (C) the nests are congruence classes for both operations $\wedge, \vee$; these congruence classes form a lattice.

Proof: In lemma 5 the HSL of the halfnests as congruence classes is commutative because ab and ba in the flat case (look at lemma 4) belong to the same halfnest.

§ 7. In this paragraph, evalueting something more about the ordered skew lattices, we often use the signs $\cdot, +$ instead of $\wedge, \vee$.

Definition: As the tolerant distributive law we denote the following axiom, consisting of two dually symmetric equations:

(73) ($D_0$) $$\begin{cases} a_\wedge(b_\vee c) = a_\wedge(b_\vee [a_\wedge c]), \\ (c_\wedge b)_\vee a = ([c_\vee a]_\wedge b)_\vee a. \end{cases}$$

This notation is reasonable because in the commutative case each line of (73) gives the usual distributive law.
For from

(74) $$a_\frown(b_\smile c) = a_\frown(b_\smile[a_\frown c])$$

we get (putting $a_\frown b$ instead of $b$) the usual modular law:

(75) $$a_\frown([a_\frown b]_\smile c) = [a_\frown b]_\smile[a_\frown c],$$

and again using (74), we get the distributive law.

The axiom (73) is a conservative HN-axiom.

Definition: The following axiom is called the modular law:

(76)(M) $$[(a_\wedge b)_\vee c]_\wedge(a_\vee b) = (a_\wedge b)_\vee[c_\wedge(a_\vee b)] \; .$$

Lemma 22: This modular law can be formulated also in the following manner: If two elements $x, y$ fulfill the relations

(77) $$\begin{cases} x_\wedge y = x, \\ x_\vee y = y \end{cases}$$

(meaning that $x$ is twofold weakly included in $y$ ), then for every element $c$ it is:

(78) $$(x_\vee c)_\wedge y = x_\vee(c_\wedge y).$$

Proof: Inserting for $x, y$ in (78) the expressions (77), we trans= form (78) into the relation (76), so that the property of modular skew lattices, formulated in (77),(78), indeed is a consequence of (76). And the elements $a_\wedge b = x$, $a_\vee b = y$ fulfil (77), so that (76) is a consequence of the law formulated in (77), (78).

The modular axiom (76) is a dually symmetrical conservative HN-axiom. The axiom (71) is a special case of (76).

Lemma 23: Any ordered skew lattice fulfils the tolerant distributive axiom (73) and the modular axiom (76).

<u>Proof:</u> The relation $a(b+c) = a(b+ac)$ is fulfilled in the case $ac = c$. In the other case $ac = a$ we have

$$a(b+c) = ac(b+c) = ac = a = a(b+a). \tag{79}$$

If $xy = x$, $x+y = y$, then in the case $x+c = x$ (and therefore $cx = x$) we get:

$$\left\{\begin{aligned} &(x+c)y = xy = x; \\ &x+cy = x+cxy = x+cx = x+c = x. \end{aligned}\right. \tag{80}$$

In the other case $x + c = c$ we have to prove $cy = x+cy$, and this is valid, if $cy = y$.

But if $cy = c$, and $x+cy = x$, therefore $x = c$, then $cy = x = x+cy$.

<u>Lemma 24: In an ordered skew lattice the axioms (C) and (F) are equivalent. They express that the ordered skew lattice is a chain composition of nests.</u>

<u>Proof:</u> In a chain composition of nests (C) and (F) are fulfilled, because they are conservative axioms, and valid in a nest. According to (F) two elements belonging to the same multiplicative halfnest cannot belong to different additional halfnests, so that one of these elements is additively strongly included in the other one. According to lemma 21 (C) has the same meaning in ordered skew lattices.

<u>Lemma 25: Any HN-axiom $\varphi(a,b) = \psi(a,b)$ valid also in $V_2$, the lattice with two elements, is fulfilled in every ordered lattice.</u>

With $\varphi(a,b)$ we denote here any well defined element of the free skew lattice with two generating elements $a,b$.

<u>Proof:</u> In an ordered skew lattice any pair of elements is a subsystem, and therefore $V_2$ or a halfnest.

# CHAPTER IV. DISTRIBUTIVE AND MODULAR SKEW LATTICES

§ 8. Naturally the tolerant distributive law (73) and the modular law (76) are not the only possibilities to generalise - in a simple manner - for the noncommutative case the distributive and the modular axiom of the commutative theory. Other possibilities will be studied in the next paragraphs.

Before doing so we at first mention:

Lemma 26: In any modular lattice the elements which are twofold weakly included in the element $y$ form a sub system.

This lemma too - similar to lemmas 22,23 - shows that (76) is a singularly simple and meaningful axiom.

Proof: If (77) and therefore (78) is fulfilled, and if $z_\wedge y = z$, we have $(x_\vee z)_\wedge y = x_\vee z$. At the other hand $x_\wedge z_\wedge y = x_\wedge z$ : the elements $x_\vee z$ and $x_\wedge z$ are multiplicativela weakly included in $y$. The rest of lemma 26 is already expressed in lemma 15.

We formulate now another distributive law:

$$(81)\ (D_\wedge)\quad \begin{cases} a_\wedge(b_\vee c) = (a_\wedge b)_\vee(a_\wedge c), \\ (c_\wedge b)_\vee a = (c_\vee a)_\wedge(b_\vee a). \end{cases}$$

Obviously (73) is a consequence of (81).

Definition: A skew lattice fulfilling the axioms (73) and (81) may be called a distributive-modular one.

The rest of this paragraph will entirely be devoted to the task to determine and to discuss the free flat distributive - modular skew lattice with two generating elements $a,b$.

Lemma 27: The free flat distributive-modular skew lattice with two generating elements $a,b$ has 18 elements. It is superflat and doubly distributive.

The term "doubly distributive" means validity of (81) and also of:

$$(82)\ (D_2)\quad \begin{cases} (a_\vee b)_\wedge c = (a_\wedge c)_\vee (b_\wedge c), \\ c_\vee (a_\wedge b) = (c_\vee a)_\wedge (c_\vee b). \end{cases}$$

Obviously (73) is a consequence of (82) too. But (81) as well as (82) is stronger than (73), for (81), (82) are not conservative axioms. But they both are HN - axioms.

The 18 elements of the skew lattice from lemma 27 - it may be denoted in the following as $W_{18}$ - are those of table 1.

TABLE 1

| | |
|---|---|
| $u_1 = a$ | $v_1 = b$ |
| $u_2 = ab$ | $v_2 = ba$ |
| $u_3 = b + a$ | $v_3 = a + b$ |
| $u_4 = ba + a$ | $v_4 = ab + b$ |
| $u_5 = a + ab$ | $v_5 = b + ba$ |
| $u_6 = b + ab$ | $v_6 = a + ba$ |
| $u_7 = ba + ab$ | $v_7 = ab + ba$ |
| $u_8 = a + b + ab$ | $v_8 = a + b + ba$ |
| $u_9 = a + ba + ab$ | $v_9 = ab + b + ba$ |

Proof: From the generating elements $a,b$ we get at first the (superflat!) free flat HSL with elements $a,b,ab,ba$. These four elements generate an additive HSL which we show to be superflat. It possesses the 18 elements of table 1.

We shall prove:

(83) $$a + ab + b = a + b$$

(84) $$a + b + ba = b + a + ba;$$

from (83) we have (by substitution of $b$ by $ba$)

(85) $$a + ab + ba = a + ba;$$

from (84) we get, substituting $ab$ instead of $a$:

(86) $$b + ab + ba = ab + b + ba.$$

From (76)(M) we have

(87) $$(a + b)a = a + ba;$$

from (81), second line:

(88) $$(a + b)b = ab + b.$$

Therefore:

(89) $$\begin{cases} a + b = (a + b)(a + b) = (a + b)a + (a + b)b \\ \quad = a + ba + ab + b = a + ab + b; \end{cases}$$

now (83) is proved to be correct.

Then from (81):

(90) $$a + b + ba = a + b(b + a);$$

from (76):

(91) $$a + b(b + a) = (a + b)(b + a) = (a + b)\,b + (a + b)a;$$

therefore from (87), (88):

(92) $$a + b + ba = ab + b + a + ba = b + a + ba$$

now (84) is proved to be correct.

Therefore <u>the additive HSL generated by $a, b, ab, ba$ indeed is superflat:</u>

We have to prove the relation $x + y + z = y + x + z$ only for the case of three different elements;

if $z = b$, only $x, y \neq ba$ is to be looked for, and (83) answers this. If $z = ba$, the three cases cleared by (84), (85), (86) are to be considered. Therefore <u>the elements of table 1 form an additive</u> (superflat) HSL.

Now we shall show that they form also a <u>multiplicative</u> HSL: It is sufficient, to show that each one of them gives another element of table 1 when multiplied with the element $a$ from the right side.

The cases $u_3$, $v_3$ are cleared by (87), (88); the modular law clears all those cases where the sum $u_k$ or $v_k$ has as its first member $a$ or $ab$. The case $v_5$ is cleared by $v_5 = bv_3$; and at last we get:

$$(93) \quad \begin{cases} u_6 = (b + a)b \quad ; \quad u_7 = (b + a)ab; \\ u_6 a = (b + a)\, ba = u_7 = u_7 a. \end{cases}$$

Here the <u>dual</u> relation to (84) has been used. <u>R E S U L T :</u> All elements of the skew lattice looked for are contained in table 1. It can also easily be seen now that both lines of the distributive law (82) are fulfilled.

In order to prove now that all these 18 polynomials in table 1 are <u>different</u> elements in our <u>free</u> system, we have to prove that they form indeed a flat distributive-modular skew lattice. After this proof, it is certain - in consequence of dual symmetry - that also the multiplicative HSL of these 18 elements is superflat.

To perform this last step of our proof we make use of the skew lattice $W_4$ described in (44). This skew lattice fulfils (76) according to lemma 23; and (81) obviously too. Now we construct the <u>direct product</u> of <u>seven</u> direct factors $W_4$; and we take from this direct product the following two elements:

$$(94) \quad \begin{cases} a = (u\,|\,u0u1\,|\,01), \\ b = (v\,|\,0u1u\,|\,10). \end{cases}$$

Showing that these elements a,b generate 18 different elements, we perform the rest of our proof. We get these 18 elements by calculating the 18 polynomials of table 1 resulting from $u_1 = a$, $v_1 = b$ according to (44); the results are summarized in table 2.

TABLE 2.

| | |
|---|---|
| $u_1 = (u\|u0u1\|01)$ | $v_1 = (v\|0u1u\|10)$ |
| $u_2 = (u\|u0uu\|00)$ | $v_2 = (v\|0uuu\|00)$ |
| $u_3 = (u\|uuu1\|11)$ | $v_3 = (v\|uu1u\|11)$ |
| $u_4 = (u\|uuu1\|01)$ | $v_4 = (v\|uu1u\|10)$ |
| $u_5 = (u\|u0uu\|01)$ | $v_5 = (v\|0uuu\|10)$ |
| $u_6 = (u\|uuuu\|10)$ | $v_6 = (v\|uuuu\|01)$ |
| $u_7 = (u\|uuuu\|00)$ | $v_7 = (v\|uuuu\|00)$ |
| $u_8 = (u\|uuuu\|11)$ | $v_8 = (v\|uuuu\|11)$ |
| $u_9 = (u\|uuuu\|01)$ | $v_9 = (v\|uuuu\|10)$ |

Lemma 28: The skew lattice $W_{18}$ of Lemma 27 can be represented by (94) as a sub system of the direct product of seven direct factors $W_4$.

Apart from helping to prove lemma 27, the representation (94) leads to further valuable information about the skew lattice $W_{18}$.

1) Introducing as a further additional axiom that one formulated in (65), we get in $W_{18}$ the congruences

(95) $$u_6 \equiv v_9; \quad u_9 \equiv v_6; \equiv \quad u_7 \equiv v_7; \quad u_8 \equiv v_8.$$

They arise from table 2 by introducing $u \equiv v$ according to (65). Therefore:

Lemma 29: The free flat halfcommutative distributive-modular skew lattice with two generating elements a,b has 14 elements. It can be represented by

(96) $$a = (u0u1|01); \quad b = (0u1u|10)$$

as a sub system of the direct product of six direct factors $W_4$.

2) Introducing additionally the axiom (68)(C) - according to lemma 24 the axiom (67)(F) would lead to the same result - we have to consider that (C) is valid in nests and in lattices, but not in a halfcommutative halfnest. Therefore in table 2 we must omit the letters between the strokes in order to get the skew lattice of the congruence classes in $W_{18}$ corresponding to (C):

(97) $$a = (u|01); \quad b = (v|10).$$

Lemma 30: The free system with two generating elements among the distributive-modular skew lattices fulfilling (C) is the direct product of two direct factors $V_2$ and one direct factor $N_2$ ( = nest with two elements).

3) Introducing the "supermodular" axiom

(98) $$x_\vee(c_\wedge y) = (x_\vee c)_\wedge y$$

in $W_{18}$ - the modular axiom (M) is a weaker consequence of (98) - into $W_{18}$ we have to omit from (94) the two direct factors $V_2$, because $V_2$ does not fulfil (98). But (98) is an HN-axiom. Therefore we get in $W_{18}$ from (98) the following congruence classes:

(99) $$\left\{\begin{array}{ll|ll} u_1 & = (u|u0u1) & v_1 & = (v|0u1u) \\ u_2 & \equiv u_5 & v_2 & \equiv v_5 \\ & \equiv (u|u0uu) & & \equiv (v|0uuu) \\ u_3 & \equiv u_4 & v_3 & \equiv v_4 \\ & \equiv (u|uuu1) & & \equiv (v|uu1u) \\ u_6 & \equiv u_7 \equiv u_8 \equiv u_9 & v_6 & \equiv v_7 \equiv v_8 \equiv v_9 \\ & \equiv (u|uuuu) & & \equiv (v|uuuu) \end{array}\right.$$

Lemma 31: The free flat supermodular distributive skew lattice with two generating elements $a = u_1$, $b = u_2$ has 8 elements. It fulfils every HN-axiom $\varphi(a,b,c,\dots) = \psi(a,b,c,\dots)$.

Proof: This $W_8$ is a sub system of a direct product of halfnests. Immediately from lemma 27 and lemma 28, together with lemma 25, we get:

**Lemma 32: All those HN-axioms $\varphi(a,b) = \psi(a,b)$ which are valid also in $V_2$, are fulfilled in all distributive-modular skew lattices.**

§ 9. We discuss now the special ordered skew lattice $W_4$ from (44). This $W_4$ and the direct products of direct factors $W_4$ fulfil a series of meanigful axioms. These we shall summarize ( as far as they are known) and then discuss their connection or indüpendencies.

0) $W_4$ is flat.

1) Every HN-axiom $\varphi(a,b) = \psi(a,b)$ which also holds in $V_2$ is valid.

2) The distributive law $(D_1)$, (81) is valid.

3) The modular law (M), (76) is valid.

4) A second modular law

(100) $$[(a_\vee b)_\wedge c]_\vee(b_\wedge a) = (a_\vee b)_\wedge[c_\vee(b_\wedge a)]$$

is valid. This is again a dually symmetric HN-axiom, but not conservative.

5) The HN-axiom

($H^*$) (101) $$\begin{cases} (b + c)(a + c)a = (b + c)a, \\ a + ca + cb = a + cb \end{cases}$$

is valid. - Proof: Its second line is fulfilled in $W_4$ in each one of the cases $c = 1$ and $c \neq 1$.
Obviously ($H^*$) is a weaker consequence of (H), (69).

6) The axiom

(102) ($C^*$) $$\begin{aligned} &(b + c)a(a + c) = (b + c)a, \\ &ca + a + cb = a + cb \end{aligned}$$

is valid. - Proof as for ($H^*$). - This is a consequence of (C),(68), and a weaker one: It is not an HN-axiom, but it is valid in every flat skew lattice which is a halfnest. -(In the following we use for such a case the denotation $HN^*$-axiom).

7) Any sub system generated by three elements $bc, ac, ab$ is doubly distributive and superflat.

<u>Proof:</u> In $W_4$ in the case $a = 1$ and in the case $a \neq 1$ this sub system is generated by only <u>two</u> elements.

8) Any sub system generated by three elements $bc, ca, ab$ is doubly distributive.

<u>Proof:</u> In $W_4$ such a sub system is generated by only two elements, if one of the elements $a, b, c$ equals $1$. In the other case it is a halfnest.

9) The following axioms are valid:

$$(b\wedge c)\vee(a\wedge b)\vee(a\wedge c) = (b\vee a)\wedge(c\vee a)\wedge(b\vee c), \tag{103}$$

$$(b\wedge a)\vee(b\wedge c)\vee(a\wedge c) = (b\vee a)\wedge(b\vee c)\wedge(a\vee c), \tag{104}$$

$$(b\wedge c)\vee(b\wedge a)\vee(a\wedge c) = (b\vee a)\wedge(a\vee c)\wedge(b\vee c), \tag{105}$$

$$(a\wedge b)\vee(b\wedge c)\vee(a\wedge c) = (b\vee a)\wedge(b\vee c)\wedge(c\vee a), \tag{106}$$

$$(c\wedge b)\vee(a\wedge b)\vee(a\wedge c) = (c\vee a)\wedge(b\vee a)\wedge(b\vee c), \tag{107}$$

$$(a\wedge b)\vee(c\wedge b)\vee(a\wedge c) = (c\vee a)\wedge(b\vee c)\wedge(b\vee a). \tag{108}$$

Each one of these six relations is a dually symmetric distributive law; (106), (108) are $HN^*$-axioms; the other four ones are HN-axioms.

10) The following $HN^*$-axiom and the dual one are valid:

$$cb + ab + ac = ab + cb + ac. \tag{109}$$

This is a special case of 7). It has the consequence that (107) and (108) are equivalent.

11) The left hand sides of (103), (104), (105), (106) are equal; and the corresponding right hand sides are equal. Therefore the four axioms (103), (104), (105), (106) are equivalent.

We write separately:

$$bc + ba + ac = ba + bc + ac; \tag{110}$$

this is an HN-axiom, and again a special case of 7).

And:

$$(111)\qquad bc + ab + ac = ab + bc + ac;$$

this is an $HN^*$-axiom, and again a special case of 7).

And:

$$(112)\qquad bc + ab + ac = bc + ba + ac .$$

This curious relation is an HN-axiom.

We now give some further remarks about the connection between these axiomatic properties of direct products of direct factors $W_4$.

A first contribution is given by lemma 32: The properties 0),2),3) have 1) as consequence. We prove now, that 0), 1), 2) lead to 4), or more precisely:

<u>Lemma 33: The distributive law ($D_1$) together with the two conservative HN-axioms</u>

$$(113)\qquad \begin{cases} (a + b)(a + b + ba) = a + b + ba; \\ (a + b + ba)(a + b)ba = (a + b + ba)ba \end{cases}$$

<u>leads to the second modular law (100).</u>

<u>Proof:</u> From ($D_1$) we have:

$$(114)\qquad \begin{cases} (a + b)(c + ba) = (a + b)c + (a + b)ba \\ \quad = \left[a + b + (a + b)ba\right]\left[c + (a + b)ba\right] \\ \quad = (a + b)\left[a + b + ba\right]\left[c + (a + b)ba\right] ; \end{cases}$$

and then from (113) and ($D_1$):

$$(115)\qquad \begin{cases} (a + b)(c + ba) \\ = \left[a + b + ba\right]\left[c + (a + b)ba\right] \\ = (a + b + ba)c + ba = (a + b)c + ba. \end{cases}$$

By quite a complicated proof the author has shown in his last paper about skew lattices:

Lemma 34: Property 7) above is a consequence of the combined axioms 0), 2), 3), 5), 6).

The proof may be omitted here. -

~~In this combination (H ) is independent. For the case (12) (12) violates (H ), but fulfils the other four axioms.~~

The property 8) has not yet been studied; it is unknown which axioms can guarantee its validity.

Under 10), 11) the four axioms (109), (110), (111), (112) and the dually corresponding ones are special cases of 7), as mentioned already above. But among these (110) can be derived already from 1),2): Using our results concerning $W_{18}$, we have

$$(116)\quad \begin{cases} bc + ba + ac = b(c + a) + ac = (b + ac)(c + a + ac) \\ = (b + ac)(a + c + ac) = b(a + c) + ac = ba + bc + ac. \end{cases}$$

Also the distributive law (103) is a consequence already from 1),2):

$$(117)\quad \begin{cases} bc + ab + ac = bc + a(b + c) \\ = (bc + a)(b + c) = (b + a)(c + a)(b + c). \end{cases}$$

The distributive law (107) is a consequence of 0), 1), 2): In a <u>flat</u> skew lattice (M) gives also

$$(118)\qquad cb + a(b + c) = (cb + a)(b + c),$$

because $cb$ is twofold weakly included in $b + c$. Similar as in (117) we come from (118) by $(D_1)$ to (107).

The axiom (112) and the dual one remain as probably independent of the other ones.

§ 10. The supermodular skew lattice $W_8$, defined by (99), is an example of a class of skew lattices which we shall study more closely in this paragraph.

Lemma 35: In the supermodular case we have

$$(119) \qquad \begin{cases} bc + ba = ba; \\ (a+b)(c+b) = a+b. \end{cases}$$

Proof: According to (98) and (2) we have

$$bc + ba = (bc + b)a = ba.$$

Lemma 36: In the supermodular case also the second modular axiom (100) is valid.

Proof: From (119) we get:

$$(a+b)c + ba = a + bc + ba = a + ba;$$
$$(a+b)(c+ba) = (a+b)(c+b)a = (a+b)a = a + ba.$$

Lemma 37: In the supermodular case each one of the axioms $(D_o)$, $(D_1)$, $(D_2)$ is equivalent to

$$(120) \qquad \begin{cases} ab + c = a + c, \\ a(b+c) = ac. \end{cases}$$

Proof: From $(D_o)$ we get now:

$$a(b+c) = a(b+ac) = a(b+a)c = ac;$$

therefore (120) is a consequence of $(D_o)$. With (119) we get ~~as of~~ From (120) also $(D_1)$. From (120) we come to $(D_2)$ thus:

$$(a+b)c = a + bc = ac + bc.$$

Lemma 38: Any superflat supermodular skew lattice is distributive.

Proof: From (119) we have in the superflat case:

$$ba + x = bc + ba + x = ba + bc + x = bc + x,$$

and with $a = b$:

$$b + x = bc + x.$$

Lemma 39: Any distributive supermodular skew lattice fulfils every HN-axiom $\varphi(a,b,c,\dots) = \psi(a,b,c,\dots)$.

Proof in the following.

Lemma 40: The skew lattices (studied above) with $a \wedge b = a \vee b$ are distributive and supermodular.

For they fulfil (120) and also the defining axiom (98) of supermodular skew lattices. Generalising this type of skew lattices - analysed in lemma 14 - we can say: Let the skew lattice $W_1$ be a multiplicative halfnest, and the skew lattice $W_2$ be an additive halfnest. Then the direct product $W_1 \times W_2$ is a distributive supermodular skew lattice, because it fulfils every HN-axiom.

Lemma 41: The free (or free flat, or free superflat) distributive supermodular skew lattice with $n$ generating elements is a sub system $U$ of the direct product of two skew lattices $W_1$, $W_2$ thus that $W_1$ is a multiplicative, and $W_2$ an additive halfnest. The additive $HSL_1$ of $W_1$ and the multiplicative $HSL_2$ of $W_2$ is the free (or free flat, or free superflat) HSL of $n$ generating elements. The sub system $U$ is the set of those elements in $W_1 \times W_2$ in which the last summand in $HSL_1$ (one of the generating elements) is the same as the first factor in $HSL_2$.

Proof: The looked for skew lattice $W$ being doubly distributive, each of its elements is an element of the additive HSL generated by the elements of the multiplicative HSL generated by the generating elements $a_1, a_2, \ldots, a_n$. But in any such sum only the last term has to contain more than one factor $a_k$ - the other ones, according to (120), can be written as single elements $a_j$. Therefore the general element $a$ of $W$ can be written as $a = \alpha' + A$, where $\alpha'$ is an element of the additive HSL generated by the $a_k$, and $A$ an element of the multiplicative HSL generated by the $a_k$. If the first factor of $A$ is $a_j$, then

$$(121) \qquad a = \alpha' + A = (\alpha' + a_j)A,$$

so that $a$ may also be written as $a = \alpha A$ where the last summand

in $\alpha$ is the same generating element $a_j$ as the first factor in $A$.

The operations $\wedge, \vee$ then take the form

$$(122)\quad \begin{cases} \alpha A + \beta B = (\alpha + \beta) B, \\ \alpha A \beta B = \alpha AB. \end{cases}$$

This proof of lemma 41 gives also the proof of lemma 39.-

Let us now construct according to lemma 41 the not flat generalisation of $W_8$.

<u>Lemma 42: The free distributive-supermodular skew lattice with two generating elements a,b has 18 elements, as given in table 3:</u>

TABLE 3.

| | |
|---|---|
| $s_1 = a$ | $t_1 = b$ |
| $s_2 = ab$ | $t_2 = ba$ |
| $s_3 = aba$ | $t_3 = bab$ |
| $s_4 = b + a$ | $t_4 = a + b$ |
| $s_5 = b + ab$ | $t_5 = a + ba$ |
| $s_6 = b + aba$ | $t_6 = a + bab$ |
| $s_7 = a + b + a$ | $t_7 = b + a + b$ |
| $s_8 = a + b + ab$ | $t_8 = b + a + ba$ |
| $s_9 = a + b + aba$ | $t_9 = b + a + bab$ |

From table 3 we come back to $W_8$ by upsetting the following congruences:

$$(123)\quad \left\{ \begin{array}{c|c} s_2 \equiv s_3 & t_2 \equiv t_3 \\ s_4 \equiv s_7 & t_4 \equiv t_7 \\ s_5 \equiv s_6 \equiv s_8 \equiv s_9 & t_5 \equiv t_6 \equiv t_8 \equiv t_9 \end{array} \right. .$$

According to lemma 41 the free distributive supermodular skew lattice with $n$ generators has $\frac{1}{n}B(n)^2$ elements, if $B(n)$ is the number of elements in the free HSL with $n$ generating elements. Therefore we get $18 = \frac{1}{2}.6^2$ elements if $n = 2$. In the case $n = 3$ we should get $\frac{1}{3}.159^2$ elements.

For free flat or superflat skew lattice of this type we get as number of elements:

$$(124) \qquad n.2^{2n-2}, \quad \text{respectively} \quad n!(n-1)!\left(\sum_{k=0}^{n-1} \frac{1}{R!}\right)^2 .$$

## CHAPTER V. CONSTRUCTION OF SKEW LATTICES FROM LATTICES.

§ 11. In any HSL -we write it here as an additive one, denoting the composition by $\cup$ - a function $fa = a'$ of the element $a$ may be defined, having the properties

$$(125) \qquad \boxed{\begin{aligned} &fa \cup a = a; \\ &f(fa \cup b) = fa \cup fb. \end{aligned}}$$

Then we get a new HSL with the same elements, but with a new composition, defined by

$$(126) \qquad a \vee b = fa \cup b.$$

Proof: From (125), (126) we have

$$(127) \qquad \begin{aligned} &a \vee a = fa \cup a = a; \\ &a \vee (b \vee c) = (a \vee b) \vee c. \end{aligned}$$

Lemma 43: The two elements $fa$ and $ffa$ form a halfnest in $W$.

Therefore in a commutative W we have

(128) $$ffa = fa.$$

Proof: From (125) we have

(129) $$\begin{cases} ffa_\cup fa = fa, \\ ffa = fa_\cup ffa. \end{cases}$$

Lemma 44: Weak $\vee$-inclusion of a in b is equivalent with weak $\cup$-inclusion of fa in fb. Weak $\cup$-inclusion of a in b has the consequence of weak $\vee$-inclusion of a in b.

Proof: $a_\vee b = b$ means $fa_\cup b = b$, therefore $f(fa_\cup b) = fb = fa_\cup fb$. At the other hand from $fb = fa_\cup fb$ we get $fa_\cup b = b$. - From $a_\cup b = b$ we have $fa_\cup b = a_\cup b = b$.

Remark: Sufficient (not necessary) conditions for the second line of (125) are:

(131) $$\begin{cases} ffa = fa, \\ f(a_\cup b) = fa_\cup fb. \end{cases}$$

**Lemma 45:** <u>In the case</u> $W = V$ (= commutative) <u>and</u> (131) <u>we have</u>

(132) $$f(a_\cap b) = f(fa_\cap b) = f(fa_\cap fb).$$

<u>Proof:</u> From Lemma 44 we have

$$f(a_\cap b) \supseteq f(fa_\cap b);$$

this together with

$$f(a_\cap b) \subseteq fa_\cap fb \subseteq fa_\cap b,$$

$$f(a_\cap b) = ff(a_\cap b) \subseteq f(fa_\cap b)$$

gives $$f(a_\cap b) = f(fa_\cap b).$$

Obviously $ffa = fa$ is a special case of (132).

<u>Lemma 46: Replacing in a skew lattice $W$ with compositions denoted by $\cap, \cup$ the composition $\cup$ by $\vee$ according to (126), with a function $fa$ having the properties (125), we get a new skew lattice $W'$ possessing the same elements as $W$, but the compositions $\cap, \vee$.</u>

<u>Proof:</u> Additionally to the remarks made above we see that replacing $\cup$ by $\vee$ we <u>loose</u> no case of additive weak inclusion (according to lemma 44), and we <u>win</u> no new case of additive strong inclusion:

$$fa_\cup b = a \rightarrow a_\cup b = a.$$

<u>Lemma 47: If the $\cup$-HSL in $W$ is flat (or even superflat), then the $\vee$-HSL in $W'$ is also flat (or even superflat).</u>

<u>Proof</u>: From the axiom $a_\cup b_\cup a = b_\cup a$ we get $a_\vee b_\vee a = fa_\cup fb_\cup a = fa_\cup fb_\cup fa_\cup a = fb_\cup fa_\cup a = fb_\cup a$. From the axiom $a_\cup b_\cup c = b_\cup a_\cup c$ we get $a_\vee b_\vee c = b_\vee a_\vee c$.

All these facts are valid in dual symmetry for $\cap, \wedge$ instead of $\cup, \vee$; and we may also replace <u>both</u> $\cap, \cup$ by $\wedge, \vee$ according to (126)

and

$$(133) \qquad a_\wedge b = a_\cap Fb$$

with

$$(134) \qquad \boxed{\begin{array}{l} F(a_\cap Fb) = Fa_\cap Fb; \\ a_\cap Fa = a. \end{array}}$$

The new skew lattice with $\wedge, \vee$ may be called $W'$.

<u>Together with $W$ the new $W'$ is flat or even superflat.</u>

Therefore we can by this construction derive from <u>commutative</u> lattices only <u>superflat</u> skew lattices, even if we make <u>repeatedly</u> such a replacement.

<u>Lemma 48: If $W$ fulfils the axiom (C), then $W'$ fulfils (C) if and only if</u>

$$(135) \qquad fFa_\cup a = Ffa_\cap a = a.$$

<u>Proof:</u> From (135) we get with (2) that

$$\begin{aligned} (b_\wedge a)_\vee a &= f(b_\cap Fa)_\cup a \\ &= f(b_\cap Fa)_\cup fFa_\cup a = f(f(b_\cap Fa)_\cup Fa)_\cup a \\ &= f(f(b_\cap Fa)_\cup (b_\cap Fa)_\cup Fa)_\cup a \\ &= f\left[(b_\cap Fa)_\cup Fa\right]_\cup a = fFa_\cup a = a. \end{aligned}$$

At the other hand any element $a$ with the property $fFa \neq a$ would give $(Fa_\wedge a)_\vee a = fFa_\cup a \neq a$.

**Lemma 49: If $W$ fulfils the axioms (C) and (M), then $W''$ too fulfils (M).**

*Proof:* It is sufficient to discuss the case $Fa = a$, which means that only replacement of $\cup$ by $\vee$ is performed. If (M) is valid in $W$, the case $x \cap y = x$ and $x \vee y = fx \cup y = y$ gives:

$$(x \vee a) \cap y = (fx \cup a) \cap y = fx \cup (a \cap y) = x \vee (a \cap y). \tag{136}$$

Now from (C) we have $fa \cap a = fa$ as consequence of $fa \cup a = a$, and therefore:

$$fx \cap y = fx \cap x \cap y = fx \cap x = fx. \tag{137}$$

**Lemma 50: If $W$ fulfils (C) and (M), then the definitions**

$$fa = a \cap s; \quad Fa = s' \cup a \tag{138}$$

**with two arbitrary constant elements $s, s'$ fulfils (125) and** (134). Therefore (138) gives then a modular skew lattice $W''$.

*Proof:* Validity of (134) is to be seen from

$$\left\{\begin{aligned} F(a \cap Fb) &= s' \cup [a \cap (s' \cup b)] \\ &= (s' \cup a) \cap (s' \cup b) = Fa \cap Fb; \end{aligned}\right. \tag{139}$$

for $s' = x$ is twofold weakly included in $s' \cup b = y$. And we have

$$a \cap Fa = a \cap (s' \cup a) = a. \tag{140}$$

**Lemma 51: In the case of a distributive lattice $W = V$ we get by (138) a skew lattice fulfilling the two distributive laws $(D_1)$ and $(D_2)$.**

Proof: Writing

$$fa = as \quad , \quad Fa = s' + a$$

we get

(140,1)
$$\begin{cases} c_\wedge [b_\vee a] = c(s' + bs + a), \\ (c_\wedge b)_\vee (c_\wedge a) = [c(s' + b)]s + c(s' + a) \\ \qquad = c(s's + bs + s' + a) \ ; \end{cases}$$

(140,2)
$$\begin{cases} [b_\vee a]_\wedge c = (bs + a)(s' + c) \ , \\ (b_\wedge c)_\vee (a_\wedge c) = b(c + s')s + a(c + s') \\ \qquad = (bs + a)\ (c + s'). \end{cases}$$

Lemma 52: If $W$ is a distributive lattice $V$, and $fF$ fulfil (131) and the dually corresponding relations

(141)
$$\begin{cases} FFa = Fa, \\ F(a_\frown b) = Fa_\frown Fb, \end{cases}$$

then $W'$ is tolerantly distributive.

Proof: We have also the dual relation to (132), which means:

(142)
$$F(a_\smile b) = F(a_\smile Fb) = F(Fa_\smile Fb).$$

Now the relation

$$c_\wedge (b_\vee a) = c_\wedge [b_\vee (c_\wedge a)]$$

wins the meaning

$$\begin{aligned} & c_\frown F(fb_\frown a) = c_\frown F[fb_\smile (c_\frown Fa)] \\ = \ & c_\frown F[(fb_\smile c)_\frown (fb_\smile Fa)] = c_\frown F(fb_\smile c)_\frown F(fb_\smile Fa) \\ = \ & c_\frown F(fb_\smile c)_\frown F(fb_\smile a); \end{aligned}$$

and this indeed is fulfilled in consequence of

$$F(fb \cup c) \supseteqq Fc \supseteqq c;$$

according to lemma 44 and the dual statements. -

At last we discuss some possibilities to construct functions $f, F$ in certain special cases of commutative $W = V$; in these cases the second line of (125) will be fulfilled in the special form (131).

<u>Construction I:</u> The lattice $V$ may consist of those pairs $a = (A_1, A_2)$ of elements $A, B, \ldots$ of a lattice $V_o$ which fulfil the condition

(143) $$A_1 \subseteqq A_2;$$

and $V$ may be a sublattice of the direct product of two direct factors $V_o$.

<u>Definition:</u>

(144) $$fa = (A_1, A_1); \; Fa = (A_2, A_2).$$

<u>Construction II.</u> Again we take a lattice $V_o = \{A, B, \ldots\}$, and we form a direct product of three direct factors $V_o$. We define $V$ as the sublattice of this direct product ~~of those~~ consisting of those tripels $a = (A_1, A_2, A_3)$ which fulfil

(145) $$A_1 \subseteqq A_2 \subseteqq A_3.$$

<u>Definition:</u>

(146) $$fa = (A_1, A_1, A_3); \; Fa = (A_1, A_3, A_3).$$

<u>In this case axiom (C) is valid</u>, according to lemma 48.

<u>Construction III</u>, including and generalising the constructions I, II: Again we take a direct product of direct factors $V_0$; the elements can also be denoted as <u>functions</u> $a = A(k)$ of an index $k$, the $A(k)$ being elements of $V_0$. In the set $M$ of index values $k$ any <u>quasi order</u> (as defined above) may be given; and we consider now the sublattice of <u>those</u> functions $A(k)$ with

$$(147) \qquad A(k) \subseteqq A(l) \quad \text{in each case} \quad k \subseteq l.$$

In the set $M$ there may be defined two functions $\varphi(k) = \varphi k$, $\phi(k) = \phi k$ with values out of $M$ fulfilling with respect to the mentioned quasi order the relations

$$(148) \qquad \varphi\varphi k = \varphi k \subseteq k \subseteq \phi k = \phi\phi k.$$

<u>Definition:</u>

$$(149) \qquad \begin{cases} (fA)(k) = A(\varphi k), \\ (FA)(k) = A(\phi k). \end{cases}$$

Sufficient for (C) is

$$(150) \qquad \varphi\phi k \subseteq k \subseteq \phi\varphi k;$$

and (150) is also necessary for (C), if $V_0$ has more than one element.

The following example includes the constructions I,II:

$$k = 1,2,\dots \quad , n;$$

$$1 \leq i,j \leq n;$$

$$\varphi k = \begin{cases} 1 \text{ if } k \subseteq i \neq k, \\ i \text{ if } i \subseteq k; \end{cases}$$

$$\phi k = \begin{cases} j \text{ if } k \subseteq j, \\ n \text{ if } j \subseteq k \neq j. \end{cases}$$

<u>Lemma 53: If $V_o$ is a distributive lattice, then construction I leads to a skew lattice $W'$ fulfilling the distributive law $(D_2)$.</u>

It may have some methodical interest to give two different proofs of this remarkable lemma.

At first we consider the simplest special case: $V_o$ may be the lattice $V_2$ of only two elements. Then $W$ is the ordered lattice of 3 elements:


We have

$$(151)\quad \begin{cases} f(0) = f(z) = 0;\ f(1) = 1, \\ F(1) = F(z) = 1;\ F(0) = 0. \end{cases}$$

That this case fulfils the relation

$$(151,1)\qquad (a_\vee b)_\wedge c = (a_\wedge c)_\vee (b_\wedge c)$$

(and the dual one), can be seen easily by direct verification.

At the same time we see that this is an <u>ordered</u> skew lattice, corresponding to the symbol

$$(152)\qquad {}_\wedge(12)\,|\,{}_\vee(21).$$

Now we proof lemma 53, <u>using the fact,</u> that every distributive lattice is a <u>sublattice of a direct product of direct factors $V_2$.</u>

Therefore the $W$ of our construction is a sublattice of another $W^*$ which may be described thus: We apply the construction I to a <u>Boolean</u> lattice ( = direct product of direct factors $V_2$). <u>Now,</u> to apply construction I to a direct product $W^* = W^{(1)} \times W^{(2)}$ means to apply it to each one of the direct factors $W^{(1)}, W^{(2)}$, getting $W^{(1)''}, W^{(2)''}$, and then forming the direct product

(153) $$W^{*''} = W^{(1)''} \times W^{(2)''} .$$

Therefore the validity of (151,1) in the case (152) means also that (151,1) is valid for $W^{*''}$ and then for $W''$.

A second proof of lemma 53, to be represented now, does not make use of the fact that each distributive lattice is a sublattice of a Boolean one. (This fact naturally allows also another proof of lemma 51).

We simply calculate, using again $\cdot, +$ instead of $\cap, \cup$ :

(154) $$\begin{cases} (a_\vee b)_\wedge c = (A_1 + B_1,\ A_1 + B_2)(C_2,\ C_2); \\ (a_\wedge c)_\vee (b_\wedge c) = (A_1C_2,\ A_1C_2) + (B_1C_2,\ B_2C_2). \end{cases}$$

Remark: If a $f,F$ -construction, applied to a skew lattice with orthogonality, fulfils

(154,1) $$\overline{fa} = F\bar{a},$$

then also

(154,2) $$\overline{a_\wedge c} = \bar{c}_\vee \bar{a}.$$

The results of this paragraph show that we can get by the $f,F$ -construction a rich material of skew lattices fulfilling the tolerant distributive law $(D_o)$ as well as the modular axiom (M). But the skew lattices constructed in this manner from commutative lattices are quite special ones in a certain respect: They all are superflat ones.

Therefore we shall proceed in the next paragraph to study another construction leading to examples which are still flat ones, but not super flat ones. The resulting new skew lattices have been partly already discussed above, for after having been detected these new examples showed themselves to be accessible also independently of the construction method of the following

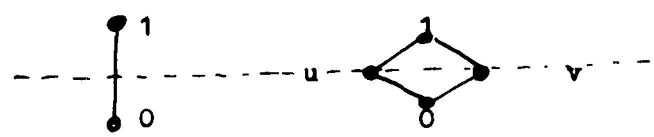
paragraph. But in spite of this the following considerations will lead us to some new aspects of the theory of skew lattices.

§ 12. At first the concept of skew lattices with <u>orthogonality</u> may be discussed a little more thoroughly.

An orthogonality $a \to \bar{a}$ with

$$(155) \quad \begin{cases} \bar{\bar{a}} = a, \\ \overline{a_\wedge c} = \bar{c}_\vee \bar{a} \end{cases}$$

exists in every lattice which cah be represented by a <u>graph</u> symmetrical to an horizontal rectilinear line. For instance:




We define $\bar{a}$ as symmetrical to $a$ . In the case of the free lattice with two generating elements $u, v$ we get thus:

$$(156) \quad \begin{cases} \bar{0} = 1; \\ \bar{u} = u, \ \bar{v} = v. \end{cases}$$

But we can also use the fact that this free lattice with two generating elements is a direct product $V_2 \times V_2$, so that the orthogonality $\bar{0} = 1$ in $V_2$ gives the orthogonality:

$$(157) \quad \begin{cases} \bar{0} = 1, \\ \bar{u} = v; \ \bar{v} = u. \end{cases}$$

In the general case of a skew lattice $W$ possessing an

orthogonality it may be $A a$ an involutorial automorphism:

$$(158)\quad \begin{cases} AAa = Aa; \\ A(a_\wedge b) = Aa_\wedge Ab, \\ A(a_\vee b) = Aa_\vee Ab; \\ A\bar{a} = \overline{Aa}. \end{cases}$$

Such an automorphism exists especially in the case that $W$ is a direct product with two isomorphic direct factors.

<u>Lemma 54: If</u> (158) <u>is fulfilled, we get a new orthogonality</u>

$a \rightarrow \tilde{a}$

<u>by the definition</u>

$$(159)\qquad \tilde{a} = A\bar{a}.$$

<u>Proof:</u> We have

$$(160)\quad \begin{cases} \tilde{\tilde{a}} = \widetilde{A\bar{a}} = A\overline{A\bar{a}} = AAa = A\ ; \\ \widetilde{a_\wedge c} = A(\overline{a_\wedge c}) = A(\bar{c}_\vee \bar{a}) = A\bar{c}_\vee A\bar{a} = \tilde{c}_\vee \tilde{a}. \end{cases}$$

Now any <u>distributive lattice</u> $V$ with operations $\cap, \cup$ and <u>with orthogonality</u> may be given, and we make the

<u>Definition:</u>

$$(161)\quad \begin{cases} a_\wedge b = a(b + \bar{a}) = ab + a\bar{a}, \\ b_\vee a = \bar{a}b + a = (a + \bar{a})(b + a). \end{cases}$$

<u>Lemma 55: With these definitions (161) the elements of $V$</u>

<u>form a skew lattice W.</u>

<u>Proof:</u> Indeed we have at first:

$$(162)\quad \left\{\begin{aligned} &(a_\wedge b)_\wedge c = (a_\wedge b)c + (a_\wedge b)(\overline{a_\wedge b}) \\ &= abc + a\bar{a}c + (ab + a\bar{a})(a\bar{b} + \bar{a}) \\ &= abc + a\bar{a}c + ab\bar{b} + a\bar{a}\bar{b} + a\bar{a}b + a\bar{a} \\ &= abc + ab\bar{b} + a\bar{a}; \end{aligned}\right.$$

and

$$(163)\quad \left\{\begin{aligned} &a_\wedge(b_\wedge c) = a(b_\wedge c) + a\bar{a} \\ &= abc + ab\bar{b} + a\bar{a}. \end{aligned}\right.$$

Therefore the compositions $\wedge, \vee$ are indeed associative ones.

Secondly we see, that replacing $\cap, \cup$ by $\wedge, \vee$ we loose no case of weak inclusion, and we win no case of strong inclusions:

$$(164)\quad \left\{\begin{aligned} &a_\cap b = a \rightarrow a_\wedge b = a; \\ &b_\vee a = b \rightarrow b_\cup a = b. \end{aligned}\right.$$

For $ab = a$ gives $a_\wedge b = a + a\bar{a} = a$; and $b_\vee a = \bar{a}b + a = b$ gives $b + a = b$.

<u>Lemma 56: W fulfils all axioms $\varphi(a,b,c,\ldots) =$ $\psi(a,b,c,\ldots)$ which are fulfilled by $W_4$.</u>

We shall see later that in all these cases W can be constructed as a sub skew lattice of a direct product of direct factors $W_4$.

Then lemma 56 is an obvious consequence. But we prefer to show here at first by direct calculation that lemma 56 is correct. According to § 9 we have to prove the following statements:

<u>W is flat.</u> For from (163) we have $a_\wedge b_\wedge a = a_\wedge b$.

<u>W fulfils (M).</u> If $x$ is twofold weakly included in $y$, we have $xy + x\bar{x} = x$, $\bar{y}x + y = y$.

Then

$$(165)\quad \left\{ \begin{aligned} x_\vee(c_\wedge y) &= \overline{(c_\wedge y)}x + (c_\wedge y) \\ &= (c\bar{y} + \bar{c})x + c(y + \bar{c}) \\ &= \bar{c}x + cy + c\bar{c}; \end{aligned} \right.$$

$$(166)\quad \left\{ \begin{aligned} (x_\vee c)_\wedge y &= (x_\vee c)y + (x_\vee c)(\bar{c}_\wedge \bar{x}) \\ &= (\bar{c}x + c)y + (\bar{c}x + c)\bar{c}(\bar{x} + c) \\ &= \bar{c}x + cy + c\bar{c}. \end{aligned} \right.$$

<u>W fulfils $(D_+)$.</u> We have

$$\begin{aligned} a_\wedge(b_\vee c) &= a\left[(b_\vee c) + \bar{a}\right] \\ &= a\left[\bar{c}b + c + \bar{a}\right]; \\ (a_\wedge b)_\vee(a_\wedge c) &= (\bar{c}_\vee \bar{a})(a_\wedge b) + (a_\wedge c) \\ &= (ac + \bar{c})a(b + \bar{a}) + a(c + \bar{a}) \\ &= a\left[(c + \bar{c})(b + \bar{a}) + c + \bar{a}\right] \\ &= a\left[\bar{c}b + c + \bar{a}\right]. \end{aligned}$$

<u>W fulfils $H^*$</u>. We have, using also (163):

$$(167)\quad \left\{ \begin{aligned} (b_\vee c)_\wedge a &= (b_\vee c)\left[a + (\bar{c}_\wedge \bar{b})\right] \\ &= (\bar{c}b + c)\left[a + \bar{c}(\bar{b} + c)\right] \end{aligned} \right.$$

(167) $$= (\bar{c}b + c)a + \bar{c}(b\bar{b} + c);$$

and

(168)
$$\begin{cases} (b_\vee c)_\wedge(a_\vee c)_\wedge a = (b_\vee c)(a_\vee c)\left[a + (\bar{c}_\wedge \bar{a})\right] + (b_\vee c)(\bar{c}_\wedge \bar{b}) \\ \quad = (\bar{c}b + c)(\bar{c}a + c)\left[a + \bar{c}(a + c)\right] + (\bar{c}b + c)\bar{c}(\bar{b} + c) \\ \quad = (\bar{c}ab + c)\left[a + \bar{c}(a + c)\right] + \bar{c}(\bar{c}b\bar{b} + c) \\ \quad = (\bar{c}b + c)a + \bar{c}(\bar{c}ab + c) + \bar{c}(b\bar{b} + c) \\ \quad = (\bar{c}b + c)\ a + \bar{c}(b\bar{b} + c). \end{cases}$$

<u>W fulfils $C^*$</u>. We have according to (163):

(169)
$$\begin{cases} (b_\vee c)_\wedge a_\wedge(a_\vee c) = (b_\vee c)a\left[(a_\vee c) + \bar{a}\right] + (b_\vee c)(\bar{c}_\vee \bar{b}) \\ = (\bar{c}b + c)a\left[\bar{c}a + c + \bar{a}\right] + (\bar{c}b + c)\bar{c}(\bar{b} + c) \\ = a(\bar{c}b\left[\bar{c} + \bar{a}\right] + c) + \bar{c}(b\bar{b} + c) \\ = (\bar{c}b + c)a + \bar{c}(b\bar{b} + c), \end{cases}$$

equal too to the expression (167).

<u>W fulfils (112).</u> From (163) we have:

(170)
$$\begin{cases} (c_\vee a)_\wedge(b_\vee a)_\wedge(c_\vee b) \\ = (c_\vee a)\left[(b_\vee a)\left[(c_\vee b) + (\bar{a}_\wedge \bar{b})\right] + (\bar{a}\ \bar{c})\right]. \end{cases}$$

Here we have:

(171)
$$\begin{cases} (b_\vee a)\left[(c_\vee b) + (\bar{a}_\wedge \bar{b})\right. \\ = (\bar{a}b + a)\left[\bar{b}c + b + \bar{a}(b + a)\right] \\ = \bar{a}b + a\left[\bar{b}c + b + \bar{a}\right]. \end{cases}$$

At the other hand:

$$
(172) \quad \begin{cases} (c_\vee a)_\wedge(a_\vee b)_\wedge(c_\vee b) \\ = (c_\vee a)\left[(a_\vee b)\left[(c_\vee b) + (\overline{b}_\wedge\overline{a})\right] + (\overline{a}_\wedge\overline{c})\right] \end{cases}
$$

with

$$
(173) \quad \begin{cases} (a_\vee b)\left[(c_\vee b) + (\overline{b}_\wedge\overline{a})\right] \\ = (\overline{b}a + b)\left[\overline{b}c + b + \overline{b}(\overline{a} + b)\right] \\ = \overline{b}ac + \overline{b}ba + \overline{b}a\overline{a} + b \\ = \overline{b}a(c + b + \overline{a}) + b. \end{cases}
$$

Therefore (170) equals:

$$
(174) \quad \begin{cases} (\overline{a}c + a)\left[\overline{a}b + a\left[\overline{b}c + b + \overline{a}\right] + \overline{a}(\overline{c} + a)\right] \\ = \overline{a}c\left[b + a + \overline{c}\right] + a\left[\overline{b}c + b + \overline{a}\right] \\ = \overline{a}c\left[b + \overline{c}\right] + a\left[\overline{b}c + b + \overline{a}\right]; \end{cases}
$$

and (172) equals

$$
(175) \quad \begin{cases} (\overline{a}c + a)\left[\overline{b}a(c + b + \overline{a}) + b + \overline{a}(\overline{c} + a)\right] \\ = \overline{a}c\left[\overline{b}a + b + \overline{c} + a\right] + a\left[\overline{b}(c + b + \overline{a}) + b + \overline{a}\right] \\ = \overline{a}c\left[b + \overline{c}\right] + a\left[\overline{b}c + b + \overline{a}\right]. \end{cases}
$$

Applied to the case (156) our definition (161) gives the skew lattice $W_4$. Therefore our proof of lemma 56 gives also a new proof of the discussed properties of $W_4$.

At last we mention still another possibility to define in a

distributive lattice with orthogonality a certain semi group. Let us consider

$$a \pm b = \bar{a}b + a\bar{b} + a\bar{a}. \tag{176}$$

a

<u>**Lemma 57: This composition (176) is an associative one.**</u>

<u>Proof</u>: At first we see that

$$(177)\quad \left\{\begin{aligned} &\overline{a \pm b} = (a + \bar{a})(b + \bar{a})(\bar{b} + a) \\ &= a(b + \bar{a}) + \bar{a}(\bar{b} + a) \\ &= a \pm \bar{b} = \bar{a} \pm b. \end{aligned}\right.$$

Therefore

$$\bar{a} \pm \bar{b} = a \pm b. \tag{178}$$

And:

$$(179)\quad \left\{\begin{aligned} &(a \pm b)(a \pm \bar{b}) \\ &= (\bar{a}b + a\bar{b} + a\bar{a})(\bar{a}b + ab + a\bar{a}) \\ &= \bar{a}b\bar{b} + ab\bar{b} + a\bar{a}. \end{aligned}\right.$$

Now we have

$$(180)\quad \left\{\begin{aligned} &a \pm (b \pm c) = \bar{a}(b \pm c) + a(b \pm \bar{c}) + a\bar{a} \\ &= \bar{a}(\bar{b}c + b\bar{c} + b\bar{b}) + a(\bar{b}\bar{c} + bc + b\bar{b}) + a\bar{a}; \end{aligned}\right.$$

$$(181)\quad \left\{\begin{aligned} (a \pm b) \pm c = {} & (a \pm \bar{b})c + (a \pm b)\bar{c} \\ & + \bar{a}b\bar{b} + ab\bar{b} + a\bar{a} \\ = {} & abc + \bar{a}\bar{b}c \\ & + a\bar{b}\bar{c} + \bar{a}b\bar{c} \\ & + \bar{a}b\bar{b} + ab\bar{b} + a\bar{a}. \end{aligned}\right.$$

From (181) we have also

(182) $a \underline{+} a \underline{+} x = a \underline{+} x \underline{+} a$

Proof: both sides of (182) equal

(183) $(a + \bar{a})x + \bar{a}a.$

Therefore

(184) $a \underline{+} a \underline{+} b \underline{+} b = a \underline{+} b \underline{+} a \underline{+} b;$

that means: The "singular" elements $a \underline{+} a$ form a sub semi group.
As one sees from (176) we have

(185) $$\begin{cases} a \underline{+} a = \bar{a}a; \\ a \underline{+} \bar{a} = \bar{a} + a. \end{cases}$$

From the expression (183), equal to (182), we learn also that

(186) $a \underline{+} a \underline{+} a = a.$

Therefore this semigroup is not an HSL, but a generalisation thereof. But the sub semi group of the singular elements $a \underline{+} a$ is an HSL.

A simple calculation shows that the definition (176) may also be written thus:

(187) $a \underline{+} b = (a_\wedge \bar{b})_\vee (\bar{a}_\wedge b).$

The constructions of this paragraph can be generalized in such a manner that instead of a distributive lattice a superflat doubly distributive skew lattice is used. But then the calculations become so awfully complicated that I prefer to omit them here.

## CHAPTER VI. DIRECT PRODUCTS OF ORDERED SKEW LATTICES

§ 13. In the theory of lattices we have the well known

Lemma 58: Each distributive lattice is a sublattice of a Boolean one.

It is the chief aim of this chapter to explore possibilities of a non commutative generalisation of this lemma. This surely is quite an hard problem. Being still far from any solution of it, I can give here only some preparatory remarks. But these already seem to show that this indeed is an highly interesting mathematical problem.

In order to get at least a well defined question, let us make the following

Definition: A skew lattice W belongs to the class D, means that it is a sub skew lattice of a direct product of ordered skew lattices.

A skew lattice W belongs to the class D', means that W has the structure of a certain system of congruence classes in a skew lattice belonging to class D.

From general considerations (P. Jordan, Abhandl. Math. Sem. Hamburg ) it is probable that the class D' can be characterised by some axioms valid in each skew lattice of type D'. How are these axioms to be found out? Surely the tolerant distributive law $(D_0)$ and the modular law (M) belong to them; but are they already sufficient?

Another question arises: Are the classes $D$ and $D'$ identical, or can we find examples of skew lattices belonging to $D'$, but not to $D$ ?

In the <u>commutative</u> case the answer is contained in lemma 58; we formulate:

<u>Lemma 59: If $V$ is a lattice of congruence classes in a lattice $V_0$ which is a sub lattice of a Boolean one, then $V$ is also equivalent to a sub lattice of</u> a direct <u>product of direct factors $V_2$.</u>

The proof of this lemma is not interesting in the frame of lattice theory, because it is only a special case of lemma. But we give here a proof which is <u>independent</u> of lemma 58.

The elements of $V_0$ may be represented as <u>functions</u> $f(x), g(x), \dots$ of $x = 1, 2, \dots, m$ with values $f = 0$ or $1$.

We have

$$(188) \quad \begin{cases} f \wedge g = fg; \\ f \vee g = f + g - fg. \end{cases}$$

Two special elements $f, g$ may be <u>congruent</u>, and we consider the system of congruences <u>generated</u> by the congruence $f \equiv g$.

The two functions $h_1(x)$, $h_2(x)$, belonging to $V_0$, may have the property that $h_1(x_0) \neq h_2(x_0)$ has the con~~gru~~sequence $f(x_0) \neq g(x_0)$. Then we have $h_1 \equiv h_2$.

For if

$$(189)\qquad \begin{cases} (h_1 \cap h_2)(x_0) = 0, \\ (h_1 \cup h_2)(x_0) = 1 \end{cases}$$

has the consequence

$$(190)\qquad \begin{cases} (f \cap g)(x_0) = 0, \\ (f \cup g)(x_0) = 1, \end{cases}$$

then for <u>all</u> values of $x$ we have:

$$(191)\qquad \begin{cases} f \cap g \cap h_1 = f \cap g \cap h_2, \\ f \cup g \cup h_1 = f \cup g \cup h_2. \end{cases}$$

Now the congruence $f \equiv g$

gives

$$(192)\qquad \begin{cases} f \cap h_1 = f \cap h_2, \\ f \cup h_1 = f \cup h_2, \end{cases}$$

and $h_1 \equiv h_2$ follows fmm the

<u>Remark</u>: In any distributive lattice from

$a \cap c = b \cap c$ and $a \cup c = b \cup c$ we have $a = b$.

<u>Proof:</u> We have

$$(193)\qquad (a \cap b) \cup (a \cap c) = a \cap (b \cup c) = a \cap (a \cup c) = a;$$

and by permutation of $a,b$ in this relation we get $a = b$.

Therefore: The congruence class of any $h_1(x)$ is determined by the values $h_1(x')$ for these $x'$ in which $f(x') = g(x')$.

Lemma 60: If a skew lattice $W$ with two generating elements $a,b$ belongs to class $D'$, then it is doubly distributive. If a flat one, it is also superflat.

Proof: In any $W$ of class $D'$ every HN-axiom

(194) $$\varphi(a,b) = \psi(ca,b)$$

valid in $V_2$ is fulfilled, as we know.

Any HN-axiom

(195) $$\Phi(x,y,z,\dots) = \Psi(x,y,z\dots)$$

valid in $V_2$, is then fulfilled in $W$. For inserting in (195) any special elements ~~in (195)~~ $x = x(a,b), y = y(a,b), \dots,$ we get

(196) $$\Phi(x(a,b),y(a,b),z(a,b),\dots) = \varphi(a,b),$$

and the validity of (195) for these $x,y,z,\dots$ is given by one of the characterised axioms (194).

Therefore $W$ is doubly distributive, so that $(D_1)$ and $(D_2)$ are fulfilled.

In the axiom

(197) $\qquad x + y + z = y + x + z$

of a superflat $W$ we again insert:

(198) $\qquad x(a,b) + y(a,b) + z(a,b) = y(a,b) + x(a,b) + z(a,b),$

getting a relation $\varphi(a,b) = \psi(a,b)$, fulfilled in the case of an additive halfnest. But ig the case of a multiplicative half-nest, <u>and a flat $W$</u>, each one of the elements $x,y,z$ reduces itself to one of the elements $a, b, a + b, b + a$; and then again (198) is fulfilled.

Knowing lemma 60, we immediately can write down the elements of the <u>free</u> skew lattice of class $D'$ with two generating elements: Thus we come to our skew lattice $W_{18}$ studied above.

<u>Lemma 61: If the class $D$ (or the class $D'$) of skew lattices can be characterised by axioms which are HN-axioms, then the following consequence is given:</u>

<u>Lemma 62 (hypothetical): Each HSL is a sub system (or equivalent to a system of congruence classes in a sub system) of a direct product of ordered skew lattices.</u>

We make a little test concerning this hypothetical lemma 62: The following statement - an extremely special case of lemma 62 - at least can be proved:

<u>Lemma 63: The free flat HSL with $n$ generating elements is a sub system of the direct product of direct factors</u> [redacted] V(12).

**Proof:** We take the direct product of $\frac{1}{2}n(n+1)$ direct factors $W_4$. Any one of the generating elements $a_k$ may be represented by a series of $\frac{1}{2}n(n+1)$ elements out of $W_4$; this series may divided into shorter series containing $n, n-1, \ldots, 1$ elements. We write:

$$(199)\quad \left\{ \begin{aligned} a_k = (&W_{k1}^{(n)} W_{k2}^{(n)} \ldots W_{kn}^{(n)} \mid W_{k1}^{(n-1)} \ldots W_{kn-1}^{(n-1)} \mid \ldots \\ &\mid W_{k1}^{(2)} W_{k2}^{(2)} \mid W_{k1}^{(1)}) \end{aligned} \right.$$

with

$$(200)\quad \left\{ \begin{aligned} &W_{kk}^{(1)} = u; \\ &W_{kj}^{(k)} = v \quad \text{if} \quad j < k; \\ &W_{rs}^{(1)} = 0 \quad \text{in all other cases.} \end{aligned} \right.$$

For instance we have, if $n = 4$:

$$a_1(u000|u00|u0|u)$$
$$a_2(0u00|0u0|vu|0)$$
$$a_3(00u0|vvu|00|0)$$
$$a_4(vvvu|000|00|0);$$

Or for $n = 5$:

$$a_1 = (u0000|u000|u00|u0|u)$$
$$a_2 = (0u000|0u00|0u0|v0|0)$$
$$a_3 = (00u00|00u0|vvu|00|0)$$
$$a_4 = (000u0|vvvu|000|00|0)$$
$$a_5 = (vvvvu|0000|000|00|0).$$

All elements generated by these $a_k$ belong to a multiplicative HSL because $0, u, v$ form a multiplicative HSL in $W_4$. Additively these $a_k$ generate a flat $\vee$-HSL with elements

$$(201)\quad \left\{ \begin{aligned} a &= a_{k_1} + a_{k_2} + \dots + a_{k_m}, \\ b &= a_{l_1} + a_{l_2} + \dots + a_{l_j}, \end{aligned} \right.$$

which are different: $a \neq b$, exactly if the corresponding elements of a <u>free</u> additive flat HSL are different, according to our discussion above. For in

$$(202)\qquad a = (Z_1^{(n)} \dots Z_n^{(n)} | Z_1^{(n-1)} \dots Z_{n-1}^{(n-1)} | \dots | Z_1^{(1)})$$

we see from the elements $Z_r^{(r)}$, <u>what</u> elements $a_{k_s}$ are contained in the sum (203) for $a$. If $a_h$ belongs to them, then $Z_h^{(h)} = u$; if not, then $Z_h^{(h)} = 0$. At the other hand, if $a_p$ and $a_q$ occur in the sum $a$, then we can see from (202), wether $a_p$ stands left and $a_q$ right, or vice versa. If $p > q$, and $a_p$ at the left side of $a_q$, then $Z_q^{(p)} = u$; otherwise $Z_q^{(p)} = v$.

It would be nice if we could now generalise lemma 63 so that for <u>all</u> flat HSL it would be shown that representation as sub system of direct products of ordered HSL must be possible - by a further step similar to lemma 59. But I cannot yet say whether this generalisation is possible. -

From the last considerations and results we gather the impression, that the tolerant distributive law $(D_o)$ alone is to weak in order to characterise - together with(M) - the class D or D'. Therefore the question arises whether there exist other distributive laws valid too in all ordered skew lattices.

A contribution to answering this question is

Lemma 64: The following relations (each one of these four lines) are conservative HN-axioms giving common distributivity in the commutative case; and they are fulfilled in every ordered skew lattice:

$$(204)\,(D_1^*)\quad \begin{cases} ca + c(a + b) + cb = ca + cb, \\ (b + c)(ba + c)(a + c) = (b + c)(a + c); \end{cases}$$

$$(205)\,(D_2^*)\quad \begin{cases} ac + (a + b)c + bc = ac + bc, \\ (c + b)(c + ba)(c + a) = (c + b)(c + a). \end{cases}$$

It is not necessary to say anything about the proof of this lemma; its correctness is obvious as soon as it has been formulated.

But the consequences of this statement are not yet known.

§ 14. The conviction that it might be possible to reduce distributive skew lattices - if properly defined axiomatically - to direct products of suitably chosen ordered skew lattices gains strong encouragement by a fact detected by W. Böge:

Lemma 65: Each skew lattice constructed from a distributive lattice according to (161), is a sub system of a direct product of direct factors $W_4$.

Proof (W. Böge): We start

from a <u>lattice</u> V. Let h be a system of exactly <u>two</u> congruence classes in V ; and H the set of <u>all</u> these h . We can describe h as a <u>function</u> of the elements x of V, possessing the values 1 and 0 according to the two congruence classes: From $h(x) = h(y) = 1$, $h(z) = h(t) = 0$ we have

$$(206)\quad \begin{cases} h(xy) = h(x + y) = h(z + x) = 1, \\ h(zt) = h(z + t) = h(zx) = 0. \end{cases}$$

With $\varphi(x)$ we denote another function of x, having as values <u>sets of elements h of</u> H, in such a manner that $\varphi(x)$ is the set of those h which fulfil $h(x) = 1$.

We ask now under which condition these subsets $\varphi$ of H form a lattice (if composed as subsets of H by $\cap, \cup$ ) which shows isomorphism to V.

<u>Lemma 66: This isomorphism is equivalent with distributivity in V.</u>

For at first it is trivial that this lattice of the $\varphi$ is distributive. But at the other hand distributivity in V is also a <u>sufficient</u> condition. Two elements $a \neq b$ of V have $\varphi(a) \neq \varphi(b)$; that means: It exists surely an h with $h(a) \neq h(b)$. One of the elements a,b - say b - may <u>not</u> be included in the other one. We take from V two subsets of elements:

$$(207)\quad \begin{cases} V_0 = \{x \text{ with } x \leqq a\}, \\ V_1 = \{x \text{ with } x \geqq b\}. \end{cases}$$

These $V_o, V_1$ are an example of pairs $U_o, U_1$ of subsets of elements of $V$ with the following properties:

$\alpha$) $\qquad U_o \supseteqq V_o$ ; $\quad U_1 \supseteqq V_1$ ,

$\beta$) $\qquad U_o \cap U_1$ = empty,

$\gamma$) $\qquad y \leqq x \in U_o \Rightarrow y \in U_o$,

$\qquad\quad y \geqq x \in U_1 \Rightarrow y \in U_1$,

$\delta$) $\qquad U_1 \cdot U_1 \leqq U_1$; $\quad U_o + U_o \leqq U_o$.

Here $\cdot, +$ may denote the compositions in the lattice $V$; we prefer here to use $\cap, \cup$ for the combinations of sets.

Assuming $V$ as finite (or otherwise using <u>Zorn's lemma</u>) we can find a <u>maximal</u> pair $U_o, U_1$; that means that from $U_o' \supseteqq U_o$, $U_1' \supseteqq U_1$ and validity of $\alpha) - \delta)$ also for $U_o', U_1'$, it follows that $U_o' = U_o$; $U_1' = U_1$.

<u>In the case of such a maximal pair $U_o, U_1$ we have</u>

(208) $\qquad U_o \cup U_1 = V.$

For if the element $c$ of $V$ would not be contained in $U_o \cup U_1$, then we have the following consequences: Let $U_1'$ be the set of those elements of $V$ which include any element of $c \cdot U_1$. We have $U_1' \supseteqq U_1$ and $U_1' \neq U_1$, because $c \in U_1'$. The pair $U_o, U_1'$ fulfils $\alpha$), $\gamma$), $\delta$), and therefore $U_o \cap U_1'$ cannot be empty.

From $\gamma$) we have then that $U_0 \cap c \cdot U_1'$ too cannot be empty; and correspondingly $U_1 \cap (c + U_0)$ cannot be empty. If now $u_0 \in U_0$, $u_1 \in U_1$ with $cu_1 \in U_0$, $c + u_0 \in U_1$, and according to $\gamma$) the element $u_0 u_1$ belongs to $U_0$, we have from $\delta$) a contradiction to $\beta$):

(209) $$u_0 u_1 + cu_1 = (u_0 + c)u_1$$

belongs to $U_0$ as well as to $U_1$. Therefore (208) is correct.

From (208) at last we see: By

(210) $$h(U_0) = 0, \; h(U_1) = 1$$

an element $h$ of $H$ with $h(a) \neq h(b)$ is defined. Therefore the proof of lemma 66 is completed.

Continuing now the proof of lemma 65 we denote by $\overline{\phantom{x}}^{\,*}$ the replacement of a subset by its complementary subset. The general case of any orthogonal correspondence in a lattice or skew lattice denoted in our former discussions by $\overline{*}$, may now be denoted by $Z$; by definition we have

(211) $$ZZ(x) = x; \;\; Z(x + y) = Z(y) . Z(x).$$

Such a $Z$ may exist in our lattice $V$; we have then, according to our former considerations, a certain permutation $\pi$ in $H$ of the order $2$ so that

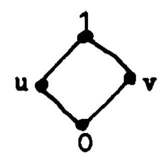
(212) $$\varphi^Z = \overline{\pi \varphi} .$$

This means that $\varphi(Z(x))$ results if one performs

the permutation $\pi$ in $\varphi(\mathfrak{X})$ and then takes the complementary set. The permutation $\pi$ obviously is the transformation

(213) $$h \longrightarrow \overline{hZ}.$$

<u>Lemma 67: Every distributive lattice with orthogonal correspondence Z can be represented as a sub system of a direct product of direct factors</u>

(214) 



<u>with</u>

(215) $$Z(u) = u, \quad Z(v) = v, \; Z(o) = 1 .$$

With the proof of this lemma 67 obviously also the proof of lemma 65 will be completed.

With respect to $\pi$ the set H consists of realms of transitivity T containing one or two elements. The lattice V is isomorphic to the lattice of the $\varphi$, which is a sublattice of the lattice P(H) of the subsets of H; and P(H) is a direct product of direct factors P(T) belonging to the different T. In the case of a T with <u>one</u> element, P(T) is equivalent to $V_2$ with $Z(o) = 1$. If T has <u>two</u> elements, P(T) is equivalent to (214), (215). - This completes our proof.

# CHAPTER VII. SUPPLEMENTS.

This chapter contains a series of additional considerations, partly scarcely connected, but contributing to the theory of skew lattices. Some of these additions here seem to show new promising paths of research, not yet explored sufficiently.

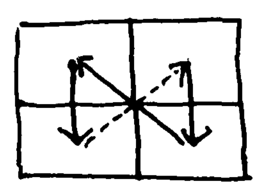
1) Definition. $A_\wedge$-HSL with the property

$$a_\wedge b_\wedge a = b_\wedge a \tag{216}$$

may be called an antiflat one.

Lemma: If a skew lattice $W$ is multiplicatively antiflat, then it must be flat additively.

Proof: Look at



The dotted arrow is a consequence of the other arrows.

2) The free HSL with $n$ generating elements is finite.

This has been shown by T.A. Green and D. Rees, Proc.Camb.Phil. Soc. 48, 35, 1952.

They proved a theorem containing this lemma as a special case. Their proof, reduced to the case interesting us, will be reported in the following.

Independently W. Böge stated and proved this theorem. His unpublished proof is not so simple as that of Green and Rees, but it contains statements which have a more general meaning and therefore may be shortly indicated here. They are apt to give important additions to the theory of skew lattices.

If two special elements $a, b$ fulfil the relation

$$bab = b, \tag{217}$$

this is equivalent to the fact that there exist $u, v$ with the property

$$uav = b. \tag{218}$$

Proof: From (218) we get

$$bav = b \quad \text{and} \quad bab = babav = bav = b.$$

The relation (217) between $a$ and $b$ is a reflexive and transitive one; writing $a|b$ we have

$$a|b,\ \&\ b|c \Longrightarrow a|c. \tag{219}$$

Proof: From $bab = b;\ cbc = c$ we get

$c = uav$ with $u = cb,\ v = bc$.

If $a|b$ and $b|a$, then we have an equivalence relation which may be denoted by $a \sim b$. The equivalence class to which an element $a$ belongs may be denoted by $\tilde{a}$.

Such an equivalence class $\tilde{a}$ is obvbusly also a sub HSL, and we know already from considerations above that it is the direct product of an halfnest and an antihalfnest. But more is to be said:

<u>Lemma : The equivalence classes a form also a system of congruence classes:</u>

(220) $$a \sim a', \; b \sim b' \Longrightarrow ab \sim a'b \sim ab'.$$

<u>Proof:</u> From $a | b$ or $bab = b$ we have, putting $u = bcb$, $v = bc$:

$$\begin{aligned} u.ac.\, v &= b.cb.a.cb.\, c \\ &= b.\, cb.\, a.\, cbab.\, c \\ &= b.\, cba.\, bc = bc.\, bc = bc; \end{aligned}$$

therefore $ac | bc$. Correspondingly (in <u>these</u> considerations strong and weak inclusion play symmetrical rôles!) we have $ca | cb$.

Our lemma 5 is the specialisation of this lemma for the flat case.

<u>Lemma : The HSL of these congruence classes $\tilde{a}$, called $H/\sim$, is commutative; and each commutative HSL of congruence classes in the original HSL is a HSL of congruence classes in $H/\sim$.</u>

<u>Proof:</u> We have

(221) $$xy.\, yx.\, xy = xy; \qquad yx | xy.$$

At the other hand $xy \mid yx$, therefore $xy \sim yx$.
And by the congruence $xy \equiv yx$ each halfnest and each antihalfnest gives only <u>one</u> congruence class.

Before continuing we indicate some considerations showing what high interest these ideas of <u>Böge's</u> are meriting.

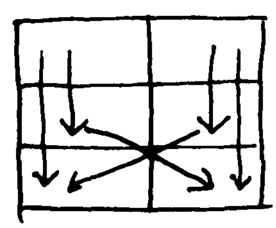
3) In words we may read $a \mid b$ thus: " <u>b is superweakly included in a</u> " in the additive case, and " <u>a is superweakly included in b</u> " in the multiplicative case.
Our graphical representation of types of inclusion may be completed thus:

(222)

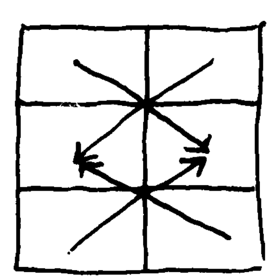
| | | |
|---|---|---|
| strong | $b_\wedge a = a$ | $b_\vee a = b$ |
| weak | $a_\vee b = b$ | $a_\wedge b = a$ |
| superweak | $a_\wedge b_\wedge a = a$ | $b_\vee a_\vee b = b$ |

.

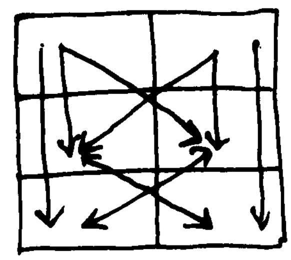
We have in the general case the consequence-relations

(223)  .

In the <u>flat</u> case we have additionally

(224)  )

so that the whole picture in the flat case is this:

(225) 

4) A new construction of HSL's from already given HSL's arises in the following manner: Let H be a HSL fulfilling the axiom (15). Then we make the definition

(226) $$a * b = aba.$$

This makes from H a new HSL, which is a flat one:

(227)
$$(a*b)*c = (a*b)c(a*b)$$
$$= abac\ aba = abcba,$$
$$a*(b*c) = abcba;$$
$$a*b*a = aba = a*b.$$

The idempotency

(228) $$a*a = aaa = a$$

(as well as the associative law) is even then fulfilled, if our starting point is not a HSL, but a more general semi group with $a^3 = a$, as we studied already above, in (186).

The associative combination $a \underline{+} b$ defined in (176) has not the property (15). But in spite of this fact even from the composition $\underline{+}$ we get by (226) an HSL. For in this case we have from (182):

$$
(229)\quad \left\{\begin{aligned}
& a \,\underline{+}\, b \,\underline{+}\, a \,\underline{+}\, c \,\underline{+}\, a \,\underline{+}\, b \,\underline{+}\, a \\
&= a \,\underline{+}\, a \,\underline{+}\, a \,\underline{+}\, a \,\underline{+}\, b \,\underline{+}\, c \,\underline{+}\, b \;=\; a \,\underline{+}\, a \,\underline{+}\, b \,\underline{+}\, c \,\underline{+}\, b; \\
& a \,\underline{+}\, b \,\underline{+}\, c \,\underline{+}\, b \,\underline{+}\, a = a \,\underline{+}\, a \,\underline{+}\, b \,\underline{+}\, c \,\underline{+}\, b.
\end{aligned}\right.
$$

Returning to the case of an HSL as the starting point of our construction, we get also

<u>**Lemma** : If in any skew lattice $W$ we replace the composition $\wedge$ by the composition $*$ according to (226), we get a new skew lattice.</u>

<u>Proof:</u> Replacing $\wedge$ by ~~get~~ $*$ we loose no case of weak inclusion, and we win no new case of strong inclusion:

$$
(230)\quad \left\{\begin{aligned}
& ab = a \Longrightarrow a * b = a; \\
& a * b = b \Longrightarrow ab = b.
\end{aligned}\right.
$$

The new classes of examples which can be constructed in this manner give an extensive new material for the study of the skew lattices. -

According to <u>Green</u> and <u>Rees</u>, also the semi groups with $x^3 = x$ have the property that the free one generated by a finite number of elements is finite.

5) The proof that the free HSL with $n$ generating elements is finite has been given by <u>Böge</u> in the continuation of his considerations presented above. Instead of following further his line of discussion we prefer only to give a sketch of the direct approach to the problem given by <u>Green</u> and <u>Rees.</u>

The element $x$ may be given by a product $X = a_{k_1} a_{k_2} \dots a_{k_r}$ of elements belonging to the generating elements $a_1, a_2, \dots, a_n$. This product may be called a word. Two words certainly correspond to the same element $x$ if they can be written as $AZB$ and $AZZB$:

(231) $$AZB \sim AZZB.$$

If it is not possible to change the word $X$ by a finite number of steps according to (231) into the word $Y$, then $Y$ represents an element $\neq x$. With $S(x) = S(X)$ we denote the set of generating elements used in any word $X$ representing $x$; obviously $S(x)$ is uniquely determined by the element $x$.

The word $X$ may have $S(X) = a_1, a_2, \dots, a_n$.
We write with other words $X^*, A, B$:

(232) $$XX = AX^* B,$$

so that

(233) $$S(A) = S(B) = S(X),$$

and so that $A, B$ have the possible minimum lengths (= number of factors in the word).

Lemma : Then $x(AX^* B) = x(AB)$;

Proof: If $X \neq A$, then

(234) $$X = X_1 a_f$$

with $a_f$ belonging to $S(X_1)$; therefore

(235) $\qquad X = Ya_f\, Y'a_f;$

and the word $XY'$ is <u>equivalent</u> to (means the same element as) the word

(236) $\qquad X_1 = Ya_f Y'$

which according to (234) is <u>shorter</u> than $X$ itself.

Therefore in (232) the word $A$ is equivalent to a certain word $XZ$:

(237) $\qquad x(A) = x(XZ).$

Now we see: The elements equivalent to words $AZ^*B$ form a <u>group</u>. Surely they form a semi group; and if $X^*, Y^*$ are given elements, we can find $Z^*$ so that

(238) $\qquad AX^*B,\ AZ^*B \sim AY^*B\ .$

For at first there exists $W$ so that

(239) $\qquad XW \sim AY^*B,$

and especially

(240) $\qquad W = XZY^*B = AX^*BZY^*B.$

In the same manner we can solve

(241) $\qquad A\bar{Z}^*B.\ AX^*B \sim AY^*B,$

so that in the semi group of elements $AY^*B$ also <u>division,</u>

right and left, is possible.

Any HSL being a group contains only <u>one element</u>. Therefore

$$x(AX^{*}B) = x(AB). \tag{242}$$

From these considerations we see that the number of elements $B(n)$ in the free HSL with $n$ generating elements is

$$\left\{ \begin{aligned} B(n) &= \sum_{k=1}^{n} \binom{n}{k} C(k); \\ C(m) &= m^2 \left[C(m-1)\right]^2; \quad C(1) = 1; \\ C(m) &= m^2 (m-1)^4 (m-2)^8 \ldots 2^{(2^{m-1})}. \end{aligned} \right. \tag{243}$$

One gets

$$B(1) = 1; \quad B(2) = 6; \; B(3) = 159; \; B(4) = 332380. \tag{244}$$

As a consequence of the theorem of <u>Green-Rees-Böge</u> we have also the following

<u>Lemma : The free doubly distributive skew lattice with $n$ generating elements is finite.</u>

But the number of its elements, certainly $\leqq B(B(n))$, must be enormous already in the case $n = 2$.

6) There are possibilities to construct special skew lattices from matrix skew rings. These possibilities are interesting, especially because they give us skew lattices with elements which are functions of continuous parameters. New types of skew lattices are to be found this way.

At first we discuss certain rings of matrices. In such a ring the axiom

$$(245) \qquad xyz = xzy$$

may be fulfilled. The general case of matrix rings with (245) is not yet known; but there exist examples which are not commutative.

The more tolerant axiom

$$(246) \qquad xy^2x + yx^2y = x^2y^2 + y^2x^2$$

is valid in all rings fulfilling (245); and also in rings fulfilling

$$(247) \qquad xyz = yxz$$

instead of (245).

Other interesting generalisations are defined by the following axioms:

$$(248) \qquad xyz + yzx + zxy = xzy + zyx + yxz;$$

$$(249) \qquad xyzt = xzyt.$$

But these cases (248), (249) will not yet be discussed here further.

In a <u>matrix ring $R$ with</u> (246) we consider the <u>idempotent</u> elements $x^2 = x$, $y^2 = y$. For these we define:

$$(250) \quad \begin{cases} x_\wedge y = xy; \\ x_\vee y = x + y - yx . \end{cases}$$

<u>The set of idempotents in $R$ form a skew lattice according to (250).</u>

<u>Proof:</u> From (246) we get now:

$$(251) \qquad xyx + yxy = xy + yx;$$

and therefore $x_\wedge y$ and $x_\vee y$ again are idempotents: Multiplying (251) with $y$ we get

$$(252) \qquad xyxy + yxy = xy + yxy;$$

Therefore $(xy)^2 = xy$; and

$$(253) \quad \begin{cases} (x_\vee y)^2 = (x + y)^2 + (yx)^2 - (yx + yxy + xyx + yx) \\ \qquad = x + y + xy - (yxy + xyx) = x + y - yx \end{cases}$$

<u>Associativity</u> of the composition $\vee$ is shown by

$$(254) \qquad x_\vee y_\vee z = x + y + z - yx - zx - zy + zyx .$$

And we have

$$(255) \quad \begin{cases} x(y_\vee x) = x(y + x - xy) = x; \\ xy_\vee x = xy + x - xy = x. \end{cases}$$

Therefore this indeed is a skew lattice; obviously the direct proof of idempotency was not necessary.

<u>Our new skew lattice is modular.</u>

<u>Proof:</u> Twofold weak inclusion of $x$ in $y$ means:

$$xy = x; \quad x_\vee y = x + y - yx = y, \tag{256}$$

or

$$xy = yx = x. \tag{257}$$

This has indeed the consequence

$$(x_\vee z)y = x_\vee zy, \tag{258}$$

or

$$(x + z - zx)y = x + zy - zyx. \tag{259}$$

This skew lattice fulfils the tolerant distributive law:

$$\left\{\begin{aligned} c\left[a_\vee cb\right] &= c\left[a + cb - cba\right] = c\left[a_\vee b\right]; \\ (b_\vee c)a_\vee c &= (b_\vee c)a + c - c(b_\vee c)a \\ &= (b + c - cb)a + c(b + c - cb)a \\ &= ba + c - cba = ba_\vee c. \end{aligned}\right. \tag{260}$$

In the more special case (245) this skew lattice fulfils

(261) $$x_\vee y_\vee x = x_\vee y.$$

It is therefore an example of the antiflat lattices discussed above, according to (216).

Proof: From (254) we have

(262) $$x_\vee y_\vee x = x + y - yx - xy + xyx,$$

and with (245) this gives

(263) $$x_\vee y_\vee x = x + y - yx = x_\vee y.$$

In this case (245) also another construction is possible:

(264) $$\begin{cases} x_\wedge y = xy, \\ x_\vee y = x + y - xy. \end{cases}$$

We then have

(265) $$\begin{cases} x(y_\vee x) = x(y + x - yx) = xy + x - xyx = x; \\ xy_\vee x = xy + x - xyx = x. \end{cases}$$

This other skew lattice too is modular.

Proof: In this case $x$ is exactly then twofold weakly included in $y$, if $xy = x$. We have then $(x_\vee z)y = x_\vee zy$ from

(266) $$(x + z - xz)y = x + zy - xzy.$$

Again the tolerant distributive law is valid:

$$(267)\quad \left\{\begin{aligned} & c\left[a_{\vee}cb\right] = ca + cb - cacb = c\left[a + b - ab\right]; \\ & (b_{\vee}c)a_{\vee}c = (b_{\vee}c)a + c - (b_{\vee}c)ac \\ & \qquad = (b + c - bc)a + c - (b + c - bc)ac \\ & \qquad = ba + c - bac = ba_{\vee}c. \end{aligned}\right.$$

This skew lattice is a flat one - other than that defined by (250), (245): For we get from (262) - a relation obviously still valid - now the consequence

$$(268)\qquad x_{\vee}y_{\vee}x = y_{\vee}x.$$

At last let us assume the existence of an element $\varepsilon$ with the property

$$(269)\qquad u\varepsilon = u$$

for all elements (not only the idempotents) of $R$. In this case we can make a curious application of the $f,F$ -construction:

$$(270)\qquad fx = Fx = \varepsilon x.$$

Here $Fx$ and $fx$ are the same function of $x$. Indeed we have

$$(271)\quad \left\{\begin{aligned} & ffx = fx = FFx = Fx = \varepsilon x; \\ & F(x_{\wedge}y) = Fx_{\wedge}Fy = \varepsilon xy; \\ & f(x_{\vee}y) = fx_{\vee}fy = \varepsilon x + \varepsilon y - \varepsilon xy; \\ & fx_{\vee}x = x; \qquad x_{\wedge}Fx = x. \end{aligned}\right.$$

The new skew lattice, resulting from the $f, F$ -construction, has

$$\text{(272)} \quad \begin{cases} x_{\wedge} y = xy, \\ x_{\vee} y = y + \xi x - \varepsilon xy. \end{cases}$$

**Appendix .**

<u>If $x^2 = x$ and $y^2 = y$, then from (248) it follows that also $z = xy = z^2$.</u>

<u>Proof:</u> From (248) we have for $z = xy$:

$$\text{(273)} \qquad 2xyxy + yxyx = xy + yxy + xyx;$$

from there:

$$2xyxy + yxyxy = xy + yxy + xyxy$$

or

$$\text{(274)} \qquad xyxy + yxyxy = xy + yxy.$$

Therefore by permutation of $x$ and $y$ and subtraction:

$$\text{(275)} \qquad xyxy - yxyx = xy - yx .$$

Adding (274) and (275) we get:

$$\text{(276)} \qquad xyxy = xy.$$

Inserting (276) in (273) we get:

$$\text{(277)} \qquad xy + yx = yxy + xyx.$$

Therefore: Also $x + y - xy$ becomes idempotent, in consequence of (248), if $x^2 = x$, $y^2 = y$.

Another consequence of (248): Replacing $x$ by $xyx$ we get:

$$yzyxy = yxyzy. \tag{278}$$

7) Another example of skew lattices: The right ideals of a semi simple skew ring with minimal chain condition form s skew lattice with respect to addition and multiplication.

8) Taking any constant element $a$ we dedine a product of $x$ and $y$ as $xay$. This gives a semi group with the property $x^3 = x^2$.

9) We study a system of 4 elements $u, v, x, y$ with the composition table

(279)

| | u | v | x | y |
|---|---|---|---|---|
| u | u | u | y | y |
| v | v | v | x | x |
| x | x | x | x | x |
| y | y | y | y | y |

,

meaning for instance that $uv = u$.

The following permutation $A$ of the elements obviously is an automorphism:

$$A = \begin{pmatrix} u & v & x & y \\ v & u & y & x \end{pmatrix}; \tag{280}$$

therefore in order to prove that (279) is <u>associative</u>, it suffices to prove the case $a(bc) = (ab)c$ with $a = u$: Indeed $u(bc) = (ub)c$ is to be verified at once for the cases $b = u, v, x, y$. Therefore (279) defines an HSL.

Now we use (279) as definition of $a_\wedge b$, and we construct $a_\vee b$ in the following manner. The permutation

$$P = \begin{pmatrix} u & v & x & y \\ x & y & v & u \end{pmatrix} \tag{281}$$

has the property

$$P^2 = A. \tag{282}$$

We define

$$a_\vee b = P\left(P^{-1}b_\wedge P^{-1}a\right), \tag{283}$$

so that we have

$$P(a_\wedge b) = Pb_\vee Pa. \tag{284}$$

<u>The definition (283) makes from the HSL (279) a skew lattice.</u>

<u>Proof:</u> The composition (283) is associative:

$$\left.\begin{aligned} (a_\vee b)_\vee c &= P\left(P^{-1}c_\wedge P^{-1}(a_\vee b)\right) \\ &= P(P^{-1}c_\wedge P^{-1}b_\wedge P^{-1}a) \\ &= a_\vee(b_\vee c). \end{aligned}\right\} \tag{285}$$

And (2) becomes equivalent to

$$(286)\quad \left\{ \begin{aligned} &Pa_{\wedge}P(a_{\wedge}b) = Pa, \\ &P^{-1}a_{\wedge}P^{-1}(a_{\wedge}b) = P^{-1}a. \end{aligned} \right.$$

In consequence of (282) these two relations are <u>equivalent</u>; and we see that in our example the first line of (286) indeed is fulfilled.

We have here a generalisation of the concept of orthogonality as discussed above. Orthogonality is the special case with A = identical permutation.

The table for the additive composition in the case of our example here obviously is:

(287)

| | u | v | x | y |
|---|---|---|---|---|
| u | u | v | u | v |
| v | u | v | u | v |
| x | u | v | x | y |
| y | u | v | x | y |

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